ALGEBRA COMPETITION 2016

請同學以題組整體完成度爲優先考量。

1. (25%) Let G be the group consisting of invertible two by two matrices with entries from a finite field $\mathbb{F}_q$.
   (a) Compute the number of conjugacy classes of G.
   (b) What are the possible cardinalities for these conjugacy classes in G?

2. (25%)
   (a) Let $\omega = \frac{-1+\sqrt{-3}}{2}$, a root of the equation $x^2 + x + 1 = 0$. Show that $\mathbb{Z}[\omega]$ is a UFD.
   (b) Let p be a prime in $\mathbb{Z}$. Show that if $p \equiv 2 \pmod 3$, then p is a prime in $\mathbb{Z}[\omega]$
   (c) Let p be a prime in $\mathbb{Z}$. Show that if $p \equiv 1 \pmod 3$, then p is not a prime in $\mathbb{Z}[\omega]$.
   (d) Is 3 a prime in $\mathbb{Z}[\omega]$? (Justify your answer)

3. (25%) Let $K = \mathbb{C}(t)$, the field of rational functions in the variable t with complex coefficients. Let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Consider the automorphism σ and τ of K over $\mathbb{C}$ defined by $\sigma(t) = t^{-1}$ and $\tau(t) = \zeta t$. Let G be the subgroup in $Aut(K/\mathbb{C})$ generated by σ and τ, and K^G be the fixed field of G.
   (a) Show that G is isomorphism to the dihedral group of order $2n$.
   (b) Compute the minimal polynomial of t over K^G.
   (c) Show that the fixed field K^G is $\mathbb{C}(u)$ for some u in $\mathbb{C}(t)$. Compute u explicitly.

4. (25%) Let A be a given finite abelian group. Let $\hat{A}$ be the set of all homomorphisms (characters) from A to the multiplicative groups of non-zero complex numbers.
   (a) Show that A and $\hat{A}$ have the same cardinality $\mid A \mid = N$.
   (b) Take N variables indexed by $a \in A$, say $\{X_a\}_{a \in A}$, and consider $\det(X_{ab^{-1}})$ as a homogeneous polynomial in these N variables. Prove the following factorization of $\det(X_{ab^{-1}})$ in $\mathbb{C}[X_a]$ as product of linear factors
   $$\det(X_{ab^{-1}}) = \prod_{\chi \in \hat{A}} \Big(\sum_{a \in A} \chi(a) X_a \Big).$$
   (c) Use (b) to write down the matrix $(X_{ab^{-1}})$ and the factorization of $\det(X_{ab^{-1}})$ explicitly in the case of $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

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(1) A K_n representation $R = (V_1, V_2, \phi_a)$ consists of finite-dimensional complex vector spaces V_1 and V_2, and n linear maps $\phi_a : V_1 \to V_2$ for $a = 1, \cdots, n$. A morphism between K_n representations (V_1, V_2, ϕ_a) and (W_1, W_2, ψ_a) consists of linear map $f_i : V_i \to W_i$ for $i = 1, 2$ such that $\psi_a f_1 = f_2 \phi_a$ for $a = 1, \cdots, n$. An isomorphism(injection) of representations is a morphism (f_1, f_2) such that f_1 and f_2 are isomorphisms(injections). Given $\theta_1, \theta_2 \in \mathbb{R}$, we define $\Theta(R) = \big(\dim(V_1)\theta_1 + \dim(V_2)\theta_2\big)/\big(\dim(V_1) + \dim(V_2)\big)$. A representation R is called Θ-semistable if for every subrepresentation R' of R we have $\Theta(R') \leq \Theta(R)$.

(a) Show that a K_2 representation $R = (\mathbb{C}^k, \mathbb{C}^k, \phi_1, \phi_2)$ is indecomposable if and only if R is isomorphic to $(\mathbb{C}^k, \mathbb{C}^k, \mathrm{Id}_k, J_\lambda)$, where J_λ is a matrix in canonical Jordan form with only one block.

(b) Show that if $\theta_1 < \theta_2$ the space of Θ-semistable K_n representations with nonzero n_1 and n_2 is empty.

(c) Show that if $\theta_1 > \theta_2$ the space of isomorphic Θ-semistable K_n representations with dimension vector $(n_1, n_2) = (1, 1)$ is $\mathbb{CP}^n$.

(d) Suppose $\theta_1 > \theta_2$ and $(n_1, n_2) = (1, m)$ with $n > m > 1$. Please construct the space of isomorphic Θ-semistable K_n representations.

(2) Let $GL_3(\mathbb{F}_2)$ be the group of 3×3 invertible matrices with entries in $\mathbb{F}_2$.

(a) Consider the action of $GL_3(\mathbb{F}_2)$ on the nonzero vectors in $\mathbb{F}_2^3$. Show that there exists an injective group homomorphism from $GL_3(\mathbb{F}_2)$ to S_7.

(b) Let A be the set of 2-dimensional subspaces in $\mathbb{F}_2^3$ and B be the set

$$B = \{\{v_1, v_2, v_3\} \mid v_i \in \mathbb{F}_2^3, \ \ v_1, v_2, v_3 \text{ are linearly independent over } \mathbb{F}_2\}.$$

Compute the cardinalities of A and B and show that $GL_3(\mathbb{F}_2)$ acts on A and B transitively.

(c) The resolvent $\Theta_g(y)$ of $g(x) \in \mathbb{Q}[x]$ of degree $n \geq 4$ with roots $\alpha_1, \cdots, \alpha_n$ is defined to be

$$\Theta_g(y) = \prod_{1 \leq i < j < k \leq n} (y - (\alpha_i + \alpha_j + \alpha_k)) \, .$$

Let $f(x) \in \mathbb{Q}[x]$ be irreducible of degree 7 and $\Theta_f(y)$ be its resolvent, which we assume to be separable. Show that if the Galois group of $f(x)$ is isomorphic to $GL_3(\mathbb{F}_2)$, then $\Theta_f(y) = f_1(y) f_2(y)$, where $f_1(y), f_2(y) \in \mathbb{Q}[y]$ are irreducible of degree 7 and 28 respectively.

(3) Let D be a division ring and D^* be the multiplicative group of non-zero elements in D.

(a) Show that if $|D| < \infty$, then D is a field. (Hint: Consider the center F of D and the class equation for D^*.)

(b) Assume $\mathrm{char}(D) = p > 0$. Show that if G is a finite subgroup of D^*, then G is cyclic.

(c) Let D be the Hamilton's real quaternions $H = \{a_0 + a_1 i + a_2 j + a_3 k \ : \ a_i \in \mathbb{R}\}$. Show that $x^2 + 1 = 0$ has infinitely many roots in D and $\sum_{i=0}^{n} a_i x^i = 0$ (where $a_1, \ldots, a_n \in \mathbb{R}$ and $a_0 \in D$ but $a_0 \notin \mathbb{R}$) has at most n roots in D.

(4) Let $I = \langle f_1, f_2 \rangle$, where $f_1 = xz - y^2$ and $f_2 = x^3 - z^2$ in $\mathbb{C}[x, y, z]$.

(a) Let $f = -4x^2y^2z^2 + y^6 + 3z^5$. Determine if $f \in I$. (Justify your answer.)

(b) Let $g = xy - 5z^2 + x$. Determine if $g \in I$. (Justify your answer.)