Examples of Smooth Solutions to Degenerate Complex Monge–Ampère Equations

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Abstract

In this paper, we will give explicit examples of solutions to the degenerate complex Monge–Ampère equation and explore some properties of the corresponding metric. Our explicit examples suggest that when the degeneration of the volume form is of conical type $|s|^{2k}, k \in \mathbb{R}^+$ for a holomorphic section of a hermitian line bundle, then the Kähler metric has at most conical singularities. Moreover, if $k \in \mathbb{N}$ (hence $|s|^{2k}$ is smooth), then the solution and therefore the degenerate Kähler metric are both smooth.

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1 Introduction

In Yau's paper [1], using the continuity method, he solved the Calabi conjecture by solving the following complex Monge–Ampère equation:

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^F\omega^n, \tag{1}$$

on a closed Kähler manifold (M^n, ω) with $F \in C^\infty(M)$ such that

$$\int_M e^F\omega^n = \mathrm{vol}(M). \tag{2}$$

Locally, Equation (1) is equivalent to

$$\det(g_{i\bar{j}} + \partial_{i\bar{j}}\varphi) = e^F \det g_{i\bar{j}}. \tag{3}$$

So we can regard this equation as prescribing the volume form of a Kähler metric $\omega + i\partial\bar{\partial}\varphi$. Hence it is related to the canonical bundle of M and the construction of Einstein metric. In [1], he showed that if $F \in C^k(M), k \geq 3$, satisfies Equation (2), then there exists $\varphi \in C^{k+1,\alpha}(M)$ for any $\alpha \in [0,1)$ such that $\omega + i\partial\bar{\partial}\varphi$ defines a Kähler metric and φ satisfies Equation (1). In particular, if $F \in C^\infty(M)$, then $\varphi \in C^\infty(M)$.

In the same paper, Yau also considered the complex Monge–Ampère equation with degenerate right-hand side:

$$(\omega + i\partial\bar{\partial}\varphi)^n = |s|^{2k}e^F\omega^n, \tag{4}$$

where s is a section of a holomorphic hermitian line bundle L such that

$$\int |s|^{2k}e^F\omega^n = \mathrm{vol}(M). \tag{5}$$

Then from the equation, we can see that on the divisor $D = \{s = 0\}$, the metric will have zero volume and hence it is degenerate here. To deal with this, Yau considered the following smoothing

$$(\omega + i\partial\bar{\partial}\varphi)^n = C_\varepsilon(|s|^2 + \varepsilon)^k e^F\omega^n, \tag{6}$$

where ε is a small positive constant and $C_\varepsilon = \mathrm{vol}(M)/\int(|s|^2 + \varepsilon)^k\omega^n$. Then by the non-degenerate case, we get a solution $\varphi_\varepsilon \in C^\infty(M)$ that satisfies Equation (6). By making some precise estimates, Yau showed that there exists a converging subsequence of $\{\varphi_\varepsilon\}$ as $\varepsilon \to 0$ in the compact set outside the divisor. Then by taking a compact resolution of $M\backslash D$, we get a solution φ of Equation (4) that is smooth outside the divisor and $|\varphi_{i\bar{j}}|$ is bounded over M for all i, j. Additionally, he also proved the uniqueness of any such solution up to a constant.

After Yau's result, some work has been done using pluripotential theory, Notably, Kolodziej had shown that on compact Käler manifold (M, ω),

$$(\omega + i\partial\bar{\partial}\varphi)^n = F\omega^n \tag{7}$$

has a continuous solution if $F \in L^1(M)$ and $\int F\omega^n = \mathrm{vol}(M)$ (Theorem 2.4.2 in [4]).

By the nature of Yau's method, we cannot know the local behavior of φ near the divisor, in particular the behavior of the degenerate metric near the divisor, which will be important for further application. For example, degeneration may occur when we pull back a Kähler metric via a birational map.

In this paper, we hope to address this problem by providing some explicit examples and try to deduce a suitable local model from them. The ultimate goal we want to prove is the following statement suggested by Prof. Chin-Lung Wang:

For $k \in \mathbb{N}$, if $F \in C^\infty(M)$ and D is a smooth divisor, then Equation (4) has a unique solution $\varphi \in C^\infty(M)$. Moreover, the degenerate metric $\omega + i\partial\bar{\partial}\varphi$ will be a conical metric transverse to the divisor, with k determining its cone angle.

And in case of $k \in \mathbb{R}_{>0}\backslash\mathbb{N}$, the solution cannot be smooth if the right hand side is not smooth. But it also won't be too bad, we should have $\varphi = f + |s|^{2(k+1)}g$ with $f, g \in C^\infty(M)$.

The term conical metric just means that near the divisor the metric is quasi-isometric to standard cone metric with cone angle $2\pi\beta$:

$$ds^2_\beta = dr^2 + \beta^2 r^2 d\theta^2 + ds^2_{\mathbb{R}^{2(n-1)}} \text{on } (\mathbb{R}^2 - \{0\}) \times \mathbb{R}^{2(n-1)}$$

where (r, θ) are polar coordinates in the first $\mathbb{R}^2$ and $ds^2_{\mathbb{R}^{2(n-1)}}$ is standard Euclidean metric. A canonical example is the pullback of the standard metric on $\mathbb{C}$ via

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & z^k. \end{array}$$

which looks like

$$ds^2 = d(z^k) \otimes d(\bar{z}^k) = k^2|z|^{2(k-1)}dz \otimes d\bar{z}.$$

in polar coordinates is

$$ds^2 = k^2 r^{2(k-1)}(dx^2 + dy^2) = k^2 r^{2(k-1)}(dr^2 + r^2 d\theta^2) = k^2 r^{2(k-1)}dr^2 + k^2 r^{2k} d\theta^2.$$

Thus if we let $\rho = r^k$, we have $d\rho = kr^{k-1}dr$. So

$$ds^2 = d\rho^2 + k^2\rho^2 d\theta^2,$$

which is the standard cone metric with cone angle $2\pi k$.

We first look at the case of one dimension in the next section, we will prove this conjecture in the smooth case using standard Hodge theory, as Equation (4) will be reduced to Laplace equation on M. We will also discuss the local behavior of solution φ near the divisor in some cases. The behavior of degenerate metric can also be read from the equation. With this we can consider the product of Riemann surfaces, which will provide many examples for both smooth and normal crossing divisors in higher dimension.

For higher dimensions in general, we hope to use the continuity method as in [1]. Hence we first have to construct some solution to Equation (4) with certain e^{F_0}. Then we start to deform the equation to the e^F we want, and show that we can also solve the equation throughout the process.

In this paper, we will construct some explicit examples that satisfy Equation (4) in $(\mathbb{C}P^n, \omega_{FS})$ and study their behavior. In Section 3, we mostly consider the case where s is a section of $L = \mathcal{O}(1)$ in $\mathbb{C}P^n$, therefore the resulting divisor is a hyperplane. With suitable metric on $O(1)$, we then show the following.

Proposition 1.1. *In $(M, \omega) = (\mathbb{C}P^n, \omega_{FS})$, for all $k \in \mathbb{N}$, if we take $\varphi_k = -\sum_{m=1}^{k} \frac{1}{m}|s|^{2m}$, then $\omega + i\partial\bar{\partial}\varphi_k$ will define a degenerate metric with vanishing order $|s|^{2k}$ along the divisor. In other word,*

$$(\omega + i\partial\bar{\partial}\varphi_k)^n = |s|^{2k} e^{F_k} \omega^n, \tag{8}$$

for some $e^{F_k} \in C^\infty(M)$.

In this example, in local coordinates $(z^1, z^2, \dots, z^n)$ on U_0 with $s = z^1$. It happens that if we write $\omega = i\partial\bar\partial\phi$, where ϕ is the local potential of Kähler metric ω. Then $\phi + \varphi_k$ has the form $f + |z^1|^{2(k+1)}h$, with $\frac{\partial}{\partial z^1} f = \frac{\partial}{\partial \bar z^1} f = 0$. Then we can see that the first column of the final metric are $g'_{i\bar 1} = (\phi + \varphi_k)_{i\bar 1} = (|z^1|^{2(k+1)}h)_{i\bar 1} = |z^1|^{2k}(\cdots)$ for $1 \le i \le n$. Hence we can factor out $|z^1|^{2k}$ from the first column, which contributes to the final degeneration of the metric in the normal direction and on the determinant. This also shows that the degenerate metric is conical transverse to D. Moreover, we have $\omega + i\partial\bar\partial\varphi_k|_D$ is another Kähler metric on the divisor, which is coming for $i\partial\bar\partial f$.

Thus, we propose an ansatz for solution φ such that the volume has degeneracy $|s|^{2k}$. When $\omega = i\partial\bar\partial\phi$, then there should be f with $f|_D$ a Käler potential such that $\phi - f \in O(|s|^{2(k+1)})$.

With this, we can further construct examples with any degree $|s|^{2k}, k \in \mathbb{R}_{>0}$. The idea is to first use the above example to create a higher vanishing order $|s|^{2m}$ for $m \in \mathbb{N}$ greater than k. Then take $\varphi = \varphi_m + |s|^{2(k+1)} f$ for suitable f, then φ will create a vanishing order of $|s|^{2k}$ on the divisor. For example, we have

Proposition 1.2. *In $(M, \omega) = (\mathbb{C}P^n, \omega_{FS})$ and $D = \{Z^1 = 0\}$. For $k \in \mathbb{N}$ and $0 < r < 1$, if we take $\varphi = \varphi_k + \frac{1}{k+r}|s|^{2(k+r)}$, where $\varphi_k = -\sum_{m=0}^{k} \frac{1}{m}|s|^{2m}$. Then $\omega + i\partial\bar\partial\varphi$ defines a Kähler metric on $M - D$ that satisfies*

$$(\omega + i\partial\bar\partial\varphi)^n = |s|^{2(k+r-1)} e^F \omega^n. \tag{9}$$

Hence we see that the solution $\varphi = \varphi_k + \frac{1}{k+r}|s|^{2(k+r)} = f + g|s|^{2(k+r)}$ as we conjectured for general degree. And again we can factor out $|s|^{k+r-1}$ from the first column of the metric, this shows that this is also conical metric as we conjectured.

In Section 4, we consider metric with degeneration on simple normal crossing divisor consisting of hyperplanes in $\mathbb{C}P^n$. We first give an example coming from the pullback of metric via the map

$$\begin{array}{ccc} \mathbb{C}P^n & \to & \mathbb{C}P^n \\ [Z^0 : \cdots : Z^n] & \mapsto & [(Z^0)^m : \cdots : (Z^n)^m] \end{array}$$

where $m \in \mathbb{N}$. Then clearly this map is locally just taking m power on each coordinate, hence the pullback metric will be conical and therefore create degeneration on divisor $\{Z^0 = 0\} \cup \cdots \cup \{Z^n = 0\}$.

Inspired by this example, we give a construction of a metric of Fubini-Study type with arbitrary vanishing order in the local chart for simple normal crossing divisor. That is, Locally in coordinates $(z^1, \dots, z^n)$, if we take

$$\omega = i\partial\bar\partial \log(1 + \sum_{i=1}^{n} |z^i|^{2(r_i+1)}).$$

Then for any $r_i \neq -1 \in \mathbb{R}$, this will give a corresponding vanishing order on $\{z^i = 0\}$. Then form this, we continue to construct global examples with different vanishing orders on all but one component.

Proposition 1.3. *On $M = \mathbb{C}P^n$. For $r_i \in \mathbb{R}_{\ge 0}$, $1 \le i \le n$, consider*

$$\omega = i\partial\bar\partial \log(|Z^0|^{2m} + \sum_{i=1}^{n} |Z^i|^{2(r_i+1)}|Z^0|^{2(m-r_i-1)} + \sum_{i=1}^{n} |Z^i|^{2m}),$$

where $m \in \mathbb{N}$ is greater than or equal to $r_i + 1$ for all $1 \le i \le n$. Then ω is well defined on the whole $\mathbb{C}P^n$ and determines a metric outside the divisor $\{Z^0 = 0\} \cup (\bigcup_{r^i > 0} \{Z^i = 0\})$, with vanishing order $|z^i|^{2r_i}$ on $\{Z^i = 0\}$ for each $i \ge 1$.

Notice that, up to a scaling $\frac{1}{m}$, this example lies in the same class as the Fubini-Study metric. Now using this construction, although we can create any vanishing order r_i on each $\{Z^i = 0\}$ for $i \neq 0$. The vanishing behavior on $\{Z^0 = 0\}$, is determined by other $\{r_i\}_{i=1}^n$. This seems to suggest some obstruction for the existence of a smooth solution with some given vanishing orders on each components, which means that the resulting φ will not be smooth, while its volume is. Nevertheless, it might be possible to twist the construction along the smooth loci of the divisors to eliminate the obstruction.

Although the divisors we consider are mostly hyperplanes in $\mathbb{C}P^n$, the local behavior may still hold in general, as any simple normal crossing divisor are locally hyperplanes. Thus, in principle, we should be able to glue our local solution near the divisor to a global one via suitable cutoff functions. In fact, this corresponds to the twisting constructions expected in the last paragraph. This is the direction we plan to work on in the sequel. With this done, it will provide a starting point for us to use the continuity method in the case of general divisor in general Kähler manifold.

Finally, in Section 5 we will set up the continuity method for the degenerate complex Monge–Ampère equation, and try to work on the openness for degenerate metric. That is, for φ_0 being a solution to

$$(\omega + i\partial\bar{\partial}\varphi_0)^n = |s|^{2k} e^{F_0} \omega^n,$$

can we find a solution to Equation (4) when F is close to F_0? We will see that we have to solve degenerate Laplace equation and develop Schauder's estimates associated with the conical metric. Some observations are presented when φ is the one we constructed in Section 3, but many progresses still need to be made in the future.

During the preparation of this thesis, I noticed a recent post [2] on arXiv by A. Bahraini where he claimed to prove that if s is a holomorphic section with simple zeros along a smooth divisor D. Then

$$(\omega + i\partial\bar{\partial}\varphi)^n = |s|^2 e^F \omega^n. \tag{10}$$

has a smooth solution.

In the sequel, based on our new constructions, we hope to generalize his result to the case with higher vanishing order as well as in the case of normal crossing divisors.

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2 Riemann surfaces and their products

2.1 complex Monge–Ampère equation on Riemann surfaces

Using standard Hodge theory, we can fully solve the complex Monge–Ampère equation in one dimension with arbitrary line bundle and divisor. Since Equation (4) in one dimension now becomes

$$\omega + i\partial\bar{\partial}\varphi = |s|^{2k} e^F \omega,$$

or locally in coordinates z,

$$g_{z\bar{z}} + \partial_{z\bar{z}}\varphi = |s|^{2k} e^F g_{z\bar{z}}.$$

Taking trace on both sides, we get

$$\Delta_g \varphi = |s|^{2k} e^F - 1. \tag{11}$$

Thus we reduce to solving this Laplace equation on M. And from the condition Equation (5), we have $\int_M (|s|^{2k} e^F - 1)\omega = 0$. By Hodge theory, this is the precise requirement for the Laplace equation to be solvable. Hence we will get a solution φ (which is unique up to a constant) for Equation (11) and multiplying the equation by $g_{z\bar{z}}$ shows that φ indeed solves the complex Monge–Ampère equation in one dimension. And by the regularity theorem, we have $\varphi \in C^\infty(M)$ if the right hand side of Equation (11) is smooth (for example, if $F \in C^\infty(M)$ and $k \in \mathbb{N}$).

The local behavior of the degenerate metric is also known, since in one dimension the determinant is exactly the same as the metric. So from equation $\omega' = \omega + i\partial\bar{\partial}\varphi = |s|^{2k} e^F \omega$, we can see that locally in coordinate z such that $z = 0$ defines the divisor, the degenerate metric is

$$|s|^{2k} e^F g_{z\bar{z}} dz \otimes d\bar{z} = |z|^{2k} e^{F'} dz \otimes d\bar{z},$$

which is a cone metric with cone angle $2\pi(k+1)$. This proves the conjecture in the one-dimensional case.

For more precise local behavior of φ, we may take chart (U, z) such that locally $s = z$, then we have φ satisfying

$$\varphi_{z\bar{z}} = |z|^{2k} e^{kh} e^F g_{z\bar{z}} - g_{z\bar{z}} = |z|^{2k} e^{F'} - g_{z\bar{z}},$$

where e^h is the local metric on L. Now by the $\partial\bar{\partial}$-lemma we can find a local potential $\varphi_g \in C^\infty(U)$ such that $(\varphi_g)_{z\bar{z}} = g_{z\bar{z}}$. Then $\varphi' = \varphi + \varphi_g$ will satisfy

$$\varphi'_{z\bar{z}} = |z|^{2k} e^{F'}, \tag{12}$$

and we now want to look at the behavior of solution φ' to this equation.

In the case where $k \in \mathbb{N}$ and $g_{z\bar{z}}, h, e^F$ are all real analytic, we have $e^{F'}$ and hence φ' are also real analytic (since we can solve Equation (12) using power series). We can therefore write $\varphi' = \sum_{i,j \geq 1} a_{ij} z^i \bar{z}^j$ and $e^{F'} = \sum_{i,j \geq 0} b_{i,j} z^i \bar{z}^j$, we omit the terms with pure power of z and $\bar{z}$ in φ' as they are harmonic and will not contribute to $\varphi'_{z\bar{z}}$. Then the equation gives

$$\varphi'_{z\bar{z}} = \sum_{i,j \geq 1} ij a_{i,j} z^{i-1} \bar{z}^{j-1} = |z|^{2k} \sum_{i,j \geq 0} b_{i,j} z^i \bar{z}^j = \sum_{i,j \geq 0} b_{i,j} z^{i+k} \bar{z}^{j+k},$$

and hence

$$\varphi' = \sum_{i,j \geq 0} \frac{b_{i,j}}{(i+k+1)(j+k+1)} z^{i+k+1} \bar{z}^{j+k+1} = |z|^{2(k+1)} \sum_{i,j \geq 0} b'_{i,j} z^i \bar{z}^j.$$

So $\varphi' = |z|^{2(k+1)} f$ for some f as we conjectured.

The more important case we want to know is the one where $k \notin \mathbb{N}$, we hope to control the behavior of φ' and therefore φ. Following our conjecture, we may try to solve $\varphi' = |z|^{2(k+1)} f$ for smooth f. Then Equation (12) becomes

$$|z|^{2k} e^{F'} = (|z|^{2(k+1)} f)_{z\bar{z}} = |z|^{2(k+1)} f_{z\bar{z}} + (k+1)|z|^{2k}(z f_z + \bar{z} f_{\bar{z}}) + (k+1)^2 f,$$

so we have

$$e^{F'} = |z|^2 f_{z\bar{z}} + (k+1)(z f_z + \bar{z} f_{\bar{z}}) + (k+1)^2 f. \tag{13}$$

Now if we assume $g_{z\bar{z}}, h, e^F$ are all real analytic like before, then $e^{F'}$ is real analytic. Let $f = \sum_{i,j\geq 0} a_{i,j} z^i \bar{z}^j$, then Equation (13) will give

$$\sum_{i,j\geq 0} (i+k+1)(j+k+1) a_{i,j} z^i \bar{z}^j = e^{F'}.$$

Since $i+k+1, j+k+1 > 0$ for all $i, j \geq 0$, we can solve this equation by comparing the coefficients on both sides.

In the future, we hope to solve Equation (13) for smooth $e^{F'}$. In general, if we can solve this for smooth f in a neighborhood of $z = 0$, then $(\varphi + \varphi_g - |z|^{2(k+1)} f)_{z\bar{z}} = 0$. Let $\psi \in C_0^\infty(U)$ be a cutoff function such that $\psi \equiv 1$ near 0, then $\Delta_g(\varphi - |z|^{2(k+1)} f\psi) \in C^\infty(M)$ and hence $\varphi - |z|^{2(k+1)} f\psi \in C^\infty(M)$ by regularity. So φ is of the form $g + |s|^{2(k+1)} f$ for some $f, g \in C^\infty(M)$ as we hoped.

We end this section with a remark on a relation between degenerate complex Monge–Ampère equation with general right hand side and the Kazdan-Warner equation about the scalar curvature under conformal change on a Riemann surface.

Remark 2.1. For Riemann surface (X, g) with Gaussian curvature K, then $\tilde{g} = e^{2u} g$ will have Gaussian curvature $\tilde{K}$ satisfying

$$\Delta_g u - K + \tilde{K} e^{2u} = 0. \tag{14}$$

And we have the following theorem due to Kazdan and Warner:

Theorem 2.2 ([5])**.** For X with genus ≥ 2, Equation (14) has a solution $u \in C^\infty(X)$ if $\tilde{K} \leq 0$ and not all 0.

Compare this with the degenerate complex Monge–Ampère equation with general right hand side in [1]:

$$(\omega + i\partial\bar{\partial}\varphi)^n = |s|^{2k} e^{F(z,\varphi)} \omega^n \tag{15}$$

with $\partial_t F(x,t) \geq 0$. In one dimension, it reduces to

$$1 + \Delta_g \varphi - |s|^{2k} e^{F(z,\varphi)} = 0,$$

and a special case of it is:

$$\Delta_g u + 1 - |s|^{2k} e^{F+2u} = 0.$$

Then these two equations coincide if our initial metric satisfies $K \equiv -1$ (which we can get by a conformal change), and we take $\tilde{K} = -|s|^{2k} e^F \leq 0$. So by the above theorem we will have a smooth solution φ in this case.

2.2 Higher dimensional examples by products

Now, the case of the Riemann surface allows us to create many examples in higher dimension by taking the product: If $\{(X_i, \omega_i)\}_{i=1}^n$ are Riemann surfaces with any metric ω_i on each X_i (which will automatically be Kähler), then we can take (M, ω) as $M = X_1 \times \cdots \times X_n$ with product metric ω. Hence if z^i is coordinate on X_i with $z^i = 0$ corresponding to $x_i \in X_i$, then $z = (z^1, \ldots, z^n)$ are coordinates on X and we have the product metric is

$$g_{i\bar{j}} = \begin{pmatrix} g_{1\bar{1}} & & \\ & \ddots & \\ & & g_{n\bar{n}} \end{pmatrix}. \tag{16}$$

Now from the above we know that if we pick x_1 as a divisor for X_1, then for any $|s|^{2k}e^F \in C^\infty(X_1)$ with $x_1 = \{s = 0\}$, we can find $\varphi_1 \in C^\infty(X_1)$ such that $\omega_1 + i\partial\bar{\partial}\varphi_1 = |s|^{2k}e^F\omega_1$. Hence the volume of $g_{i\bar{j}} + (\varphi_1)_{i\bar{j}}$ will be $g'_{1\bar{1}} \prod_{i\neq 1} g_{i\bar{i}}$, which is degenerate on the smooth divisor $\{x_1\} \times X_2 \times \cdots \times X_n$.

Similarly we can find φ_i on each X_i such that $g'_{i\bar{i}} = g_{i\bar{i}} + (\varphi_i)_{i\bar{i}}$ will have degeneration on x_i. Then

$$\begin{aligned} g'_{i\bar{j}} &= g_{i\bar{j}} + (\sum_i \varphi_i)_{i\bar{j}} \\ &= \begin{pmatrix} g_{1\bar{1}} + (\varphi_1)_{1\bar{1}} & & \\ & \ddots & \\ & & g_{n\bar{n}} + (\varphi_n)_{n\bar{n}} \end{pmatrix} = \begin{pmatrix} g'_{1\bar{1}} & & \\ & \ddots & \\ & & g'_{n\bar{n}} \end{pmatrix} \end{aligned}$$

will have volume $\prod_i g'_{i\bar{i}}$ which is degenerate on simple normal crossing divisor $D = \bigcup_i (\prod_{j<i} X_j \times \{x_i\} \times \prod_{j>i} X_j)$ which locally is just $\{\prod_i z^i = 0\}$. This provides many examples that satisfy the conjecture and obviously generalizes to the case where we choose the divisor on each X_i to consist of arbitrary many points on each X_i.

3 Examples for smooth divisor in $\mathbb{C}P^n$

For general n, we consider the simplest case that $M^n = \mathbb{C}P^n = \{[Z^0 : Z^1 : \cdots : Z^n]\}$ with Fubini-Study metric on it. We can take $U_i = \{Z^i \neq 0\} = \{[\frac{Z_0}{Z_i} : \cdots : 1 : \cdots : \frac{Z_n}{Z_i}]\}$ as charts. Then on U_0 with coordinates $(z^1, \ldots, z^n) \mapsto [1 : z^1 : \cdots : z^n]$, the Fubini-Study metric is

$$\begin{aligned} \omega_{FS} &= i\partial\bar{\partial}\log|Z|^2 = i\partial\bar{\partial}\log(1 + \sum |z_i|^2) \\ &= i\partial(\frac{z_j d\bar{z}^j}{1 + |z|^2}) = i\frac{\delta_{ij}(1 + |z|^2) - \bar{z}^i z^j}{(1 + |z|^2)^2} dz^i \wedge d\bar{z}^j. \end{aligned}$$

Hence $g_{i\bar{j}} = \frac{\delta_{ij}(1+|z|^2) - \bar{z}^i z^j}{(1+|z|^2)^2}$ and $\det g_{i\bar{j}} = \left(\frac{1}{1+|z|^2}\right)^{n+1}$.

For the divisor D, we choose the hyperplane $\{Z^1 = 0\} \simeq \mathbb{C}P^{n-1}$, which is a smooth divisor and locally on U_0 is just $\{z^1 = 0\}$. In terms of Cartier divisor, it is $\{(1, U_1), (\frac{Z^1}{Z_i}, U_i)\}$, which gives $g_{ij} = \frac{f_i}{f_j} = \frac{Z^j}{Z^i}$. This corresponds to the line bundle $\mathcal{O}(1)$, since we can look at the tautological line bundle $\mathcal{O}(-1)$. Then for (f_i, U_i) a section of $\mathcal{O}(-1)$, we get

$$f_0 \left(1, \frac{Z^1}{Z^0}, \ldots, \frac{Z^n}{Z^0}\right) = f_1 \left(\frac{Z^0}{Z^1}, 1, \ldots, \frac{Z^n}{Z^1}\right).$$

Hence we get $f_0 = f_1 \frac{Z^0}{Z^1}$ and the transition function is $g_{ij} = \frac{Z^i}{Z^j}$. Also on $\mathcal{O}(-1)$ we have a natural bundle metric h^* induced from $\mathbb{C}^{n+1}$, which on U_i is $h_i^* = \sum_{j=0}^n |\frac{Z^j}{Z^i}|^2$. Then we can take the dual metric h on $\mathcal{O}(1)$ which gives $h_i = \frac{1}{\sum_{j=0}^n |\frac{Z^j}{Z^i}|^2}$ on U_i. Then for the section $s = \{(1, U_1), (\frac{Z^1}{Z_i}, U_i)\} \in \Gamma(\mathcal{O}(1))$, we have the norm is

$$|s|^2 = h_0 \left|\frac{Z^1}{Z^0}\right|^2 = \frac{\left|\frac{Z^1}{Z^0}\right|^2}{\sum_{j=0}^n \left|\frac{Z^j}{Z^0}\right|^2} = \frac{|Z^1|^2}{\sum_j |Z^j|^2} = \frac{|z^1|^2}{1 + |z|^2}.$$

For the following two sections, we fix $(M, \omega, s, |s|^2)$ as above, and most calculations will be made on chart U_0 with standard coordinates.

3.1 Hyperplane with degeneracy $|s|^{2k}, k \in \mathbb{N}$

Our main example for this section is:

Proposition 3.1. *For all $k \in \mathbb{N}$, consider $\varphi_k = -\sum_{m=1}^{k} \frac{1}{m}|s|^{2m} \in C^\infty(M)$. Then $\omega + i\partial\bar{\partial}\varphi_k$ defines a Kähler metric on $M - D$ which satisfies*

$$(\omega + i\partial\bar{\partial}\varphi_k)^n = |s|^{2k} e^{F_k} \omega^n, \tag{17}$$

with $e^{F_k} = (k+1)(\sum_{m=0}^{k} |s|^{2m})^{n-1} > 0$.

Hence, we will have an example with smooth φ for any vanishing order $|s|^{2k}$, $k \in \mathbb{N}$ on the divisor.

To calculate $\omega' = \omega + i\partial\bar{\partial}\varphi_k$, we first need the following lemma:

Lemma 3.2. On chart U_0, we have

$$(|s|^{2k})_{i\bar{j}} = k^2|s|^{2(k-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] - k|s|^{2k} \left[\frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2} \right].$$

Proof. From

$$\begin{aligned}
(|s|^2)_{i\bar{j}} &= \left(\frac{|z^1|^2}{1+|z|^2} \right)_{i\bar{j}} = \left(\frac{\delta_{1i}\bar{z}^1}{1+|z|^2} - \frac{|z^1|^2\bar{z}^i}{(1+|z|^2)^2} \right)_{\bar{j}} \\
&= \left(\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} - \frac{|z^1|^2\delta_{ij} + \delta_{1j}z^1\bar{z}^i}{(1+|z|^2)^2} + 2\frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right) \\
&= -\frac{|z^1|^2\delta_{ij}}{(1+|z|^2)^2} + \frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + 2\frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3},
\end{aligned}$$

we have

$$\begin{aligned}
(|s|^{2k})_{i\bar{j}} &= (k|s|^{2(k-1)}|s|_i^2)_{\bar{j}} = k|s|^{2(k-1)}|s|^2_{i\bar{j}} + k(k-1)|s|^{2(k-2)}|s|_i^2|s|_{\bar{j}}^2 \\
&= -k|s|^{2(k-1)} \left[\frac{|z^1|^2\delta_{ij}}{(1+|z|^2)^2} - \frac{\delta_{1i}\delta_{1j}}{1+|z|^2} + \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} - 2\frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&\quad + k(k-1)|s|^{2(k-2)} \left(\frac{\delta_{1i}\bar{z}^1}{1+|z|^2} - \frac{|z^1|^2\bar{z}^i}{(1+|z|^2)^2} \right) \left(\frac{\delta_{1j}z^1}{1+|z|^2} - \frac{|z^1|^2 z^j}{(1+|z|^2)^2} \right) \\
&= -k|s|^{2(k-1)} \left[\frac{|z^1|^2\delta_{ij}}{(1+|z|^2)^2} - \frac{\delta_{1i}\delta_{1j}}{1+|z|^2} + \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} - 2\frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&\quad + k(k-1)|s|^{2(k-2)} \left[\frac{\delta_{1i}\delta_{1j}|z^1|^2}{(1+|z|^2)^2} - \frac{|z^1|^2\delta_{1j}z^1\bar{z}^i + |z^1|^2\delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^3} + \frac{|z^1|^4\bar{z}^i z^j}{(1+|z|^2)^4} \right] \\
&= -k|s|^{2(k-1)} \left[\frac{|z^1|^2\delta_{ij}}{(1+|z|^2)^2} - \frac{\delta_{1i}\delta_{1j}}{1+|z|^2} + \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} - 2\frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&\quad + k(k-1)|s|^{2(k-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&= k^2|s|^{2(k-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&\quad - k|s|^{2(k-1)} \left[\frac{|z^1|^2\delta_{ij}}{(1+|z^2|)^2} - \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&= k^2|s|^{2(k-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3} \right] - k|s|^{2k} \left[\frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2} \right].
\end{aligned}$$

□

Hence for $\varphi_k = -\sum_{m=1}^{k} \frac{1}{m}|s|^{2m}$, we have

$$
\begin{aligned}
(\varphi_k)_{i\bar{j}} &= -\sum_{m=1}^{k} \frac{1}{m}(|s|^{2m})_{i\bar{j}} \\
&= -\sum_{m=1}^{k} m|s|^{2(m-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3}\right] \\
&\quad + \sum_{m=1}^{k} |s|^{2m} \left[\frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2}\right]
\end{aligned}
$$

and therefore

$$
\begin{aligned}
g_{i\bar{j}} + (\varphi_k)_{i\bar{j}} &= \frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2} + (\varphi_k)_{i\bar{j}} \\
&= -\sum_{m=1}^{k} m|s|^{2(m-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2\bar{z}^i z^j}{(1+|z|^2)^3}\right] \\
&\quad + \sum_{m=0}^{k} |s|^{2m} \left[\frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2}\right] \\
&= (\sum_{m=0}^{k} |s|^{2m})\frac{\delta_{ij}}{1+|z|^2} - (\sum_{m=0}^{k}(m+1)|s|^{2m})\frac{\bar{z}^i z^j}{(1+|z|^2)^2} \\
&\quad - (\sum_{m=1}^{k} m|s|^{2(m-1)}) \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j}z^1\bar{z}^i + \delta_{1i}\bar{z}^1 z^j}{(1+|z|^2)^2}\right] \\
&= \begin{pmatrix} \frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2) & -\frac{(k+1)|s|^{2k}\bar{z}^1}{(1+|z|^2)^2}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \\ -\frac{(k+1)|s|^{2k}z^1}{(1+|z|^2)^2}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix} & \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2} I_{n-1} - \frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \end{pmatrix}.
\end{aligned}
$$

To find the determinant of this metric, we use the following lemma:

Lemma 3.3.

$$\det(A + uv^T) = \det \cdot A(1 + v^T A^{-1} u)$$

Proof. For determinant, we have

$$\det(A + uv^t) = \det A \cdot \det(I + A^{-1}uv^t) = \det A \cdot (1 + v^t A^{-1} u).$$

Where $\det(I + uv^t) = 1 + v^t u$ is coming from

$$\begin{pmatrix} I & 0 \\ v^t & 1 \end{pmatrix}\begin{pmatrix} I + uv^t & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & 0 \\ -v^t & 1 \end{pmatrix} = \begin{pmatrix} I & u \\ 0 & 1 + v^t u \end{pmatrix}.$$

□

We can now expand with respect to the first column, and get

$$\det(g') = \frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2)\det\left(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2} I_{n-1} - \frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix}\right)$$

$$+\sum_{i=2}^{n}(-1)^{i+2}\frac{(k+1)|s|^{2k}z^1\bar z^i}{(1+|z|^2)^2}\det\begin{pmatrix} -\frac{(k+1)|s|^{2k}\bar z^1}{(1+|z|^2)^2}\begin{pmatrix} z^2 & \cdots & z^n\end{pmatrix} \\ \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}I_{n-1}-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix}\bar z^2\\ \vdots \\ \bar z^n\end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n\end{pmatrix}\end{pmatrix}_{\text{-}i\text{row}}$$

$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2)(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-1}(1-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)\sum_{m=0}^{k}|s|^{2m}}(|z|^2-|z^1|^2))$$

$$+\sum_{i=2}^{n}(-1)^{i+2}\frac{(k+1)|s|^{2k}z^1\bar z^i}{(1+|z|^2)^2}\det\begin{pmatrix} -\frac{(k+1)|s|^{2k}\bar z^1}{(1+|z|^2)^2}\begin{pmatrix} z^2 & \cdots & z^n\end{pmatrix} \\ \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}I_{n-1}\end{pmatrix}_{\text{- }i\text{ row}}$$

$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-1}((1-|s|^2)-\frac{(1-|s|^2)\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)\sum_{m=0}^{k}|s|^{2m}}(|z|^2-|z^1|^2))$$

$$+\sum_{i=2}^{n}(-1)^{i+2}\frac{(k+1)|s|^{2k}z^1\bar z^i}{(1+|z|^2)^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}(-1)^{i+1}\frac{(k+1)|s|^{2k}\bar z^1 z^i}{(1+|z|^2)^2}$$

$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-1}((1-|s|^2)-\frac{\sum_{m=0}^{k}|s|^{2m}-(k+1)|s|^{2(k+1)}}{(1+|z|^2)\sum_{m=0}^{k}|s|^{2m}}(|z|^2-|z^1|^2))$$

$$-(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}(\frac{(k+1)|s|^{2k}}{(1+|z|^2)^2})^2\sum_{i=2}^{n}|z^1|^2|z^i|^2$$

$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-1}\left[1-|s|^2-\frac{|z|^2-|z^1|^2}{1+|z|^2}+\frac{(k+1)|s|^{2(k+1)}(|z|^2-|z^1|^2)}{(1+|z|^2)\sum_{m=0}^{k}|s|^{2m}}\right]$$

$$-\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}\frac{(k+1)|s|^{2k}}{(1+|z|^2)^3}|z^1|^2(|z|^2-|z^1|^2)$$

$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-1}\left[1-\frac{|z|^2}{1+|z|^2}+\frac{(k+1)|s|^{2(k+1)}(|z|^2-|z^1|^2)}{(1+|z|^2)\sum_{m=0}^{k}|s|^{2m}}\right]$$

$$-\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}\frac{(k+1)|s|^{2(k+1)}}{(1+|z|^2)^2}(|z|^2-|z^1|^2)$$

$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}$$

$$\left[\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}(\frac{1}{1+|z|^2}+\frac{(k+1)|s|^{2(k+1)}(|z|^2-|z^1|^2)}{(1+|z|^2)\sum_{m=0}^{k}|s|^{2m}})-\frac{(k+1)|s|^{2(k+1)}}{(1+|z|^2)^2}(|z|^2-|z^1|^2)\right]$$

$$=\frac{(k+1)|s|^{2k}}{(1+|z|^2)^3}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}$$

$$\left[\sum_{m=0}^{k}|s|^{2m}+(k+1)|s|^{2(k+1)}(|z|^2-|z^1|^2)-(k+1)|s|^{2(k+1)}(|z|^2-|z^1|^2)\right]$$

$$=\frac{(k+1)|s|^{2k}}{(1+|z|^2)^3}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-2}\sum_{m=0}^{k}|s|^{2m}=\frac{(k+1)|s|^{2k}}{(1+|z|^2)^2}(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2})^{n-1}.$$

Which gives $e^F=(k+1)(\sum_{m=0}^{k}|s|^{2m})^{n-1}>0$ (since $\det g_{i\bar j}=(\frac{1}{1+|z|^2})^{n+1}$).

For positive definiteness, we can see that the determinant of first $\ell \times \ell$ block is

$$\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^k |s|^{2m}}{1+|z|^2})^{\ell-1}\left[1-|s|^2-\frac{\sum_{i=2}^\ell |z^i|^2}{1+|z|^2}+\frac{(k+1)|s|^{2(k+1)}(\sum_{i=2}^\ell |z^i|^2)}{(1+|z|^2)\sum_{m=0}^k |s|^{2m}}\right]$$
$$-\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^k |s|^{2m}}{1+|z|^2})^{\ell-2}\frac{(k+1)|s|^{2k}}{(1+|z|^2)^3}|z^1|^2(\sum_{i=2}^\ell |z^i|^2)$$
$$=\frac{(k+1)|s|^{2k}}{1+|z|^2}(\frac{\sum_{m=0}^k |s|^{2m}}{1+|z|^2})^{\ell-1}\left[1-|s|^2-\frac{\sum_{i=2}^\ell |z^i|^2}{1+|z|^2}\right]$$
$$=\frac{(k+1)|s|^{2k}}{(1+|z|^2)^2}(\frac{\sum_{m=0}^k |s|^{2m}}{1+|z|^2})^{\ell-1}\left[1+|z|^2-\sum_{i=1}^\ell |z^i|^2\right],$$

which is positive outside the divisor for all ℓ. Hence from linear algebra we know that $g' = g+\partial\bar{\partial}\varphi_k$ is positive definite outside the divisor on chart U_0. (Similarly for U_i, $\forall i \neq 1$ by symmetry.)

To check that it only vanishes on the divisor, it remains to look at the chart U_1 (mainly for the point $[0 : 1 : 0 : \cdots]$ that is not cover by $\cup_{i\neq 1} U_i$). And we can without lost of generality exchange 0 and 1, then

$$|s|^2 = \frac{|Z^0|^2}{\sum |Z^j|^2} = \frac{1}{1+|z|^2},$$

and

$$|s|^2_{i\bar{j}} = \left(\frac{-\bar{z}^i}{(1+|z|^2)^2}\right)_{\bar{j}} = \frac{-\delta_{ij}}{(1+|z|^2)^2} + \frac{2\bar{z}^i z^j}{(1+|z|^2)^3},$$
$$(|s|^{2k})_{i\bar{j}} = (k|s|^{2(k-1)}|s|^2_i)_{\bar{j}} = k|s|^{2(k-1)}|s|^2_{i\bar{j}} + k(k-1)|s|^{2(k-2)}|s|^2_i|s|^2_{\bar{j}}$$
$$= k|s|^{2(k-1)}[\frac{-\delta_{ij}}{(1+|z|^2)^2} + \frac{2\bar{z}^i z^j}{(1+|z|^2)^3}] + k(k-1)|s|^{2(k-2)}\frac{\bar{z}^i z^j}{(1+|z|^2)^4}$$
$$= k|s|^{2k}[\frac{-\delta_{ij}}{1+|z|^2} + (k+1)\frac{\bar{z}^i z^j}{(1+|z|^2)}].$$

Hence

$$\varphi_k = -\sum_{m=1}^k \frac{1}{m}|s|^{2m} = -\sum_{m=1}^k \frac{1}{m}\left(\frac{1}{1+|z|^2}\right)^{2m}$$

and

$$g_{i\bar{j}} + (\varphi_k)_{i\bar{j}} = \frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2} + \sum_{m=1}^k |s|^{2m}[\frac{\delta_{ij}}{1+|z|^2} - (m+1)\frac{\bar{z}^i z^j}{(1+|z|^2)}]$$
$$= (\sum_{m=0}^k |s|^{2m})\frac{\delta_{ij}}{1+|z|^2} - (\sum_{m=0}^k (m+1)|s|^{2m})\frac{\bar{z}^i z^j}{(1+|z|^2)^2}.$$

So the determinant of first $\ell \times \ell$ block is:

$$\left(\frac{\sum_{m=0}^k |s|^{2m}}{1+|z|^2}\right)^\ell \left(1-\frac{\sum_{m=0}^k (m+1)|s|^{2m}}{\sum_{m=0}^k |s|^{2m}}\frac{\sum_{i=1}^\ell |z^i|^2}{1+|z|^2}\right).$$

Now

$$1-\frac{\sum_{m=0}^k (m+1)|s|^{2m}}{\sum_{m=0}^k |s|^{2m}}\frac{\sum_{i=1}^\ell |z^i|^2}{1+|z|^2} \geq 1-\frac{\sum_{m=0}^k (m+1)|s|^{2m}}{\sum_{m=0}^k |s|^{2m}}\frac{|z|^2}{1+|z|^2},$$

with

$$1-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{\sum_{m=0}^{k}|s|^{2m}}\frac{|z|^2}{1+|z|^2}=1-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{\sum_{m=0}^{k}|s|^{2m}}(1-|s|^2)$$
$$=1-\frac{\sum_{m=0}^{k}|s|^{2m}-(k+1)|s|^{2(k+1)}}{\sum_{m=0}^{k}|s|^{2m}}$$
$$=\frac{(k+1)|s|^{2(k+1)}}{\sum_{m=0}^{k}|s|^{2m}}>0\quad(\because |s|^2>0).$$

Hence the determinant is positive for all ℓ and in particular

$$\det(g'_{i\bar{j}})=\left(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}\right)^{n}\frac{(k+1)|s|^{2(k+1)}}{\sum_{m=0}^{k}|s|^{2m}}=\left(\frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}\right)^{n-1}\frac{(k+1)|s|^{2k}}{(1+|z|^2)^2},$$

which also gives $e^F=(k+1)(\sum_{m=0}^{k}|s|^{2m})^{n-1}$.

3.1.1 Local behavior along D

On chart U_0, since the degenerate metric is

$$g_{i\bar{j}}+(\varphi_k)_{i\bar{j}}=\begin{pmatrix}\frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2) & -\frac{(k+1)|s|^{2k}\bar{z}^1}{(1+|z|^2)^2}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\\ -\frac{(k+1)|s|^{2k}z^1}{(1+|z|^2)^2}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix} & \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}I_{n-1}-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\end{pmatrix}.$$

We see that we can factor out $|s|^{2k}$ from the first column, which contributes to the final $|s|^{2k}$ in the determinant. This also tells us that if we consider $\omega+i\partial\bar{\partial}(\varphi_k+|s|^{2(k+1)}f)$, then since $(|s|^{2(k+1)}f)_{i\bar{j}}$ will have a factor $|s|^{2k}$ for all i,j, we can still factor out $|s|^{2k}$ from the first column, and hence the result determinant will still have $|s|^{2k}$ in it, in this way we can create many possible solutions. (This may increase the order, like if we take $f=-\frac{1}{k+1}$, then we will get φ_{k+1} and hence $|s|^{2(k+1)}$ on the divisor.)

Also, the factor $|s|^{2k}=|z^1|^{2k}(\cdots)$ makes it a cone metric in the normal direction (i.e. the z^1 direction). If we take local coordinate change $z^1=w^{\frac{1}{k+1}}$, then the metric in $(w,z^2,\ldots,z^n)$ will be

$$\tilde{g}_{i\bar{j}}=\begin{pmatrix}\frac{w^{\frac{-k}{k+1}}\bar{w}^{\frac{-k}{k+1}}}{(k+1)^2}\frac{(k+1)|z^1|^{2k}}{(1+|z|^2)^{k+1}}(1-|s|^2) & -\frac{w^{\frac{-k}{k+1}}}{k+1}\frac{(k+1)|z^1|^{2k}\bar{z}^1}{(1+|z|^2)^{k+2}}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\\ -\frac{\bar{w}^{\frac{-k}{k+1}}}{k+1}\frac{(k+1)|z^1|^{2k}z^1}{(1+|z|^2)^{k+2}}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix} & \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}I_{n-1}-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}\frac{1}{(k+1)(1+|z|^2)^{k+1}}(1-|s|^2) & -\frac{\bar{w}}{(1+|z|^2)^{k+2}}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\\ -\frac{w}{(1+|z|^2)^{k+2}}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix} & \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}I_{n-1}-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\end{pmatrix},$$

with $|z^1|^2=(|w|^2)^{\frac{1}{k+1}}$ and $|s|^2=\frac{|w|^{\frac{2}{k+1}}}{1+|w|^{\frac{2}{k+1}}+\sum_{i\geq 2}|z^i|^2}$. We can see that $\tilde{g}_{i\bar{j}}$ is well defined in a neighborhood of $\{w=0\}$ and is nondegenerate here and we may regard $g'_{i\bar{j}}$ as the pullback of $\tilde{g}_{i\bar{j}}$ under the map

$$(z^1,z^2,\ldots)\to(w=(z^1)^{k+1},z^2,\ldots).$$

The factor $|s|^{2k}$ also gives a holomorphic vector field $X = \partial_{z^1}$ near the divisor that satisfies

$$\langle X, Y \rangle_{g'} = O(|s|^{2k}), \ \forall |Y|_g \leq 1.$$

Finally, on the divisor $\{Z^1 = 0\}$ the metric becomes

$$\begin{pmatrix} 0 & 0 \ \cdots \ 0 \\ 0 & \\ \vdots & \frac{1}{1+|z|^2} I_{n-1} - \frac{1}{(1+|z|^2)^2} \bar{z}^i z^j \\ 0 & \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_{FS}|_{\mathbb{C}P^{n-1}} \end{pmatrix}.$$

Hence the restriction of the degenerate metric on the divisor will be another Kähler metric.

3.1.2 Possible explanation for φ_k

From above discussion we may guess that the degeneration of determinant will comes from the first column. Hence we look at the term $\varphi_{1\bar{1}}$ on chart U_0, then from Lemma 3.2 we get

$$\begin{aligned}
(|s|^{2k})_{1\bar{1}} &= k^2|s|^{2(k-1)} \left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2} - \frac{\delta_{1j} z^1 \bar{z}^i + \delta_{1i} \bar{z}^1 z^j}{(1+|z|^2)^2} + \frac{|z^1|^2 \bar{z}^i z^j}{(1+|z|^2)^3} \right] \\
&\quad - k|s|^{2k} \left[\frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2} \right] \\
&= k^2|s|^{2(k-1)} \left[\frac{1}{1+|z|^2} - \frac{2|z^1|^2}{(1+|z|^2)^2} + \frac{|z^1|^4}{(1+|z|^2)^3} \right] \\
&\quad - k|s|^{2k} \left[\frac{1}{1+|z|^2} - \frac{|z^1|^2}{(1+|z|^2)^2} \right] \\
&= \frac{k|s|^{2(k-1)}}{1+|z|^2} \left[k - (2k+1)|s|^2 + (k+1)|s|^4 \right].
\end{aligned}$$

And since

$$\log(1+|z|^2)_{1\bar{1}} = \left(\frac{z^1}{1+|z|^2} \right)_1 = \frac{1}{1+|z|^2} - \frac{|z^1|^2}{(1+|z|^2)^2} = \frac{1}{1+|z|^2}(1-|s|^2),$$

if we assume $\varphi_k = \sum_{m=1}^k a_m |s|^{2m}$, then

$$\begin{aligned}
g_{1\bar{1}} + (\varphi_k)_{1\bar{1}} &= \frac{1}{1+|z|^2}(1-|s|^2) + \sum a_m \frac{|s|^{2(m-1)}}{1+|z|^2}[m^2 - m(2m+1)|s|^2 + m(m+1)|s|^4] \\
&= \frac{1}{1+|z|^2}(1 - |s|^2 + \sum a_m m^2 |s|^{2(m-1)} - \sum a_m m(2m+1)|s|^{2m} + \sum a_m m(m+1)|s|^{2(m+1)}) \\
&= \frac{1}{1+|z|^2}\Big(1 + a_1 + [-1 + 4a_2 - 3a_1]|s|^2 \\
&\quad + \sum_{m \geq 2} [a_{m+1}(m+1)^2 - a_m m(2m+1) + a_{m-1}(m-1)m]|s|^{2m} \Big).
\end{aligned}$$

So for $\varphi_{1\bar{1}}$ to have order $|s|^{2k}$ at the divisor, we need $a_1 = -1, a_2 = -\frac{1}{2}$ and $a_m = -\frac{1}{m}$ for $m \leq k$, which is what we define φ_k to be.

More precisely, if we let $|w|^2 = \sum_{i>1} |z^i|^2$, then the potential for the Fubini-Study metric is

$$
\begin{aligned}
\log(1+|z^1|^2+|w|^2) &= \log(1+|w|^2) + \log(1+\frac{|z^1|^2}{1+|w|^2}) \\
&= \log(1+|w|^2) - (-\frac{|z^1|^2}{1+|w|^2} + \frac{1}{2}(\frac{-|z^1|^2}{1+|w|^2})^2 + \cdots),
\end{aligned}
$$

where we use $\log(1-x) = -\sum_{m\geq 1} \frac{1}{m} x^m$. And we have

$$
\begin{aligned}
|s|^2 = \frac{|z^1|^2}{1+|z^1|^2+|w|^2} &= \frac{|z^1|^2}{1+|w|^2} \cdot \frac{1}{1+\frac{|z^1|^2}{1+|w|^2}} \\
&= \frac{|z^1|^2}{1+|w|^2}(1 - \frac{|z^1|^2}{1+|w|^2} + (\frac{-|z^1|^2}{1+|w|^2})^2 + \cdots),
\end{aligned}
$$

so

$$
\begin{aligned}
\log(1+|z|^2+|w|^2) - |s|^2 &= \log(1+|w|^2) + \frac{1}{2}(\frac{|z^1|^2}{1+|w|^2})^2 + \cdots \\
&= \log(1+|w|^2) + |z^1|^2(\cdots)
\end{aligned}
$$

and

$$
\log(1+|z^1|^2+|w|^2) - \sum_{m=1}^{k} \frac{1}{m}|s|^{2m} = \log(1+|w|^2) + |z^1|^{2(k+1)}(\cdots).
$$

With $|w|^2$ being independent of z^1, $g'_{1\bar{j}}$ will naturally have vanishing order $|z^1|^{2k}$ for all j. We can also see that the term $\log(1+|w|^2)$ is the one that contributes to the restrict metric on the divisor.

Thus we conjecture that under suitable coordinates $(z,w) \in \mathbb{C}\times\mathbb{C}^{n-1}$ such that locally the divisor is just $\{z=0\}$, the local model for the potential of degenerate metric with vanishing order $|z|^{2k}$ is of the form

$$
\varphi = f + |z|^{2(k+1)} g,
$$

where $\partial_z f = \partial_{\bar{z}} f = 0$. However this local model is quite fragile, as it will not be preserved under general coordinate change, even if we fix z. For example in $\mathbb{C}P^2$, under coordinate change $(w^1, w^2) = (z^1, z^2 + z^1)$, the potential for will become

$$
\log(1+|z^2|^2) + |z^1|^2(\cdots) = \log(1+|w^2 - w^1|^2) + |w^1|^2(\cdots).
$$

So for further application, we either need to find suitable coordinates for the local model or need to describe it in a more coordinate-free way in the future.

As a final remark, since the construction in 3.1 works for all $k \in \mathbb{N}$, it is interesting to see what will happen if we let k approach infinity. As $k \to \infty$, we get

$$
\varphi_\infty = -\sum_{m=1}^{\infty} \frac{1}{m}(|s|^2)^m = \log(1-|s|^2),
$$

which is well defined when $|s|^2 < 1$, so $\omega + i\partial\bar{\partial}\varphi_\infty$ is well defined on $\mathbb{C}P^n \backslash \{p\}$ with $p = [0:1:\cdots]$. Then this should give a vanishing order of ∞. In practice, we see that on U_0,

$$
\omega + i\partial\bar{\partial}\varphi_\infty = i\partial\bar{\partial}\log(1+|z|^2) + i\partial\bar{\partial}\log(1 - \frac{|z^1|^2}{1+|z|^2}) = i\partial\bar{\partial}\log(1+|z|^2 - |z^1|^2).
$$

With $1+|z|^2-|z^1|^2=1+\sum_{i\geq 2}|z^i|^2$ not depending on z^1, the metric $g_{i\bar{j}}$ will vanish if i or j is 1. And in the remaining direction, it becomes the Fubini-Study metric for $(z^2,\dots,z^n)$. It looks like we collapse the whole $\mathbb{C}P^n\backslash\{p\}$ into the divisor.

For the behavior near p, we exchange 0 and 1 like before. Then on chart U_0, p is just the origin, and we have

$$\omega+i\partial\bar{\partial}\varphi_\infty=i\partial\bar{\partial}\log(\sum_{i>0}|Z^i|^2)=i\partial\bar{\partial}\log(|z|^2).$$

Also from 3.1, the volume for $\omega+i\partial\bar{\partial}\varphi_k$ is $|s|^{2k}e^{F_k}\omega^n$, with

$$|s|^{2k}e^{F_k}=|s|^{2k}(k+1)(\sum_{m=0}^{k}|s|^{2m})^{n-1}=(k+1)|s|^{2k}(\frac{1-|s|^{2(k+1)}}{1-|s|^2})^{n-1}.$$

Thus if we let $f_k=|s|^{2k}e^{F_k}\in C^\infty(\mathbb{C}P^n)$, then we have $\int(f_k-1)\omega^n=0$ for all $k\in\mathbb{N}$ and

$$f_k(z)\to\begin{cases}0 & \text{if } z\neq 0\\ \infty & \text{if } z=0.\end{cases}$$

So the volume will concentrate at the point p as $k\to\infty$.

3.2 Hyperplane with degeneracy $|s|^{2k}$, $k\in\mathbb{R}_{>0}$

We now give a example with vanishing order $|s|^{2k}$ for all $k\in\mathbb{R}_{>0}$, our main result for this section is:

Proposition 3.4. *For $k\in\mathbb{N}$ and $0\leq r<1$, if we take $\varphi=\varphi_k+\frac{1}{k+r}|s|^{2(k+r)}$, where $\varphi_k=-\sum_{m=0}^{k}\frac{1}{m}|s|^{2m}$ as in Section 3. Then $\omega+i\partial\bar{\partial}\varphi$ defines a Kähler metric on $M-D$ that satisfies*

$$(\omega+i\partial\bar{\partial}\varphi)^n=|s|^{2(k+r-1)}e^F\omega^n, \tag{18}$$

with

$$e^F=\left(\sum_{m=0}^{k}|s|^{2m}-|s|^{2(k+r)}\right)^{n-1}[(k+1)|s|^{2(1-r)}+(k+r)-(k+r+1)|s|^2].$$

The discussion in Section 3.1.2 suggests that to get degree $k+r$ with $k\in\mathbb{Z}_{\geq 0}, 0<r<1$, we should consider $\varphi_{k+1}+C_0|s|^{2(k+r+1)}$, then $g_{1\bar{1}}+\varphi_{1\bar{1}}=|s|^{2(k+1)}(\dots)+|s|^{2(k+r)}(\dots)=|s|^{2(k+r)}(\dots)$ will have vanishing order $|s|^{2(k+r)}$.

For ease of notation and computation, we may take $\varphi=\varphi_k+C_0|s|^{2(k+r)}$ and $C_0=\frac{1}{k+r}$. Then from Lemma 3.2, the metric will be

$$g_{i\bar{j}}+\varphi_{i\bar{j}}=\begin{pmatrix}\frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2) & -\frac{(k+1)|s|^{2k}\bar{z}^1}{(1+|z|^2)^2}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\\ -\frac{(k+1)|s|^{2k}z^1}{(1+|z|^2)^2}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix} & \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|z|^2}I_{n-1}-\frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix}\bar{z}^2\\ \vdots\\ \bar{z}^n\end{pmatrix}\begin{pmatrix}z^2 & \cdots & z^n\end{pmatrix}\end{pmatrix}$$
$$+C_0(k+r)^2|s|^{2(k+r-1)}\left[\frac{\delta_{1i}\delta_{1j}}{1+|z|^2}-\frac{\delta_{1j}z^1\bar{z}^i+\delta_{1i}\bar{z}^1z^j}{(1+|z|^2)^2}+\frac{|z^1|^2\bar{z}^iz^j}{(1+|z|^2)^3}\right]$$
$$-C_0(k+r)|s|^{2(k+r)}\left[\frac{\delta_{ij}}{1+|z|^2}-\frac{\bar{z}^iz^j}{(1+|z|^2)^2}\right]$$

$$
= \begin{pmatrix} \frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2) & -\frac{(k+1)|s|^{2k}\bar{z}^1}{(1+|z|^2)^2}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \\ -\frac{(k+1)|s|^{2k}z^1}{(1+|z|^2)^2}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix} & \frac{\sum_{m=0}^k |s|^{2m}}{1+|z|^2} I_{n-1} - \frac{\sum_{m=0}^k (m+1)|s|^{2m}}{(1+|z|^2)^2}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \end{pmatrix}
$$

$$
+ C_0 \begin{pmatrix} (|s|^{2(k+r)})_{1\bar{1}} & \frac{|s|^{2(k+r-1)}}{(1+|z|^2)^2}[(k+r)^2(-1+|s|^2)+(k+r)|s|^2]\bar{z}^1\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \\ \ldots & -\frac{(k+r)|s|^{2(k+r)}}{1+|z|^2} I_{n-1} + \frac{|s|^{2(k+r)}}{(1+|z|^2)^2}[(k+r)^2+(k+r)]\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \end{pmatrix}
$$

$$
= \begin{pmatrix} A & B\bar{z}^1\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \\ Bz^1\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix} & CI_{n-1} + E\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \end{pmatrix}
$$

with

$$
\begin{aligned}
A &= \frac{(k+1)|s|^{2k}}{1+|z|^2}(1-|s|^2) + \frac{|s|^{2(k+r-1)}}{1+|z|^2}(1-|s|^2)[(k+r)-(k+r+1)|s|^2] \\
&= \frac{1-|s|^2}{1+|z|^2}[(k+1)|s|^{2k} + (k+r)|s|^{2(k+r-1)} - (k+r+1)|s|^{2(k+r)}] \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
B &= -\frac{(k+1)|s|^{2k}}{(1+|z|^2)^2} + \frac{|s|^{2(k+r-1)}}{(1+|z|^2)^2}[(k+r)(-1+|s|^2)+|s|^2] \\
&= -\frac{(k+1)|s|^{2k}}{(1+|z|^2)^2} - \frac{|s|^{2(k+r-1)}}{(1+|z|^2)^2}[(k+r)-(k+r+1)|s|^2] \\
&= -\frac{A}{(1-|s|^2)(1+|z|^2)}, \\
C &= \frac{\sum_{m=0}^k |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2} > 0 \quad (\because |s|^2 < 1), \\
E &= \frac{(k+r+1)|s|^{2(k+r)} - \sum_{m=0}^k (m+1)|s|^{2m}}{(1+|z|^2)^2}.
\end{aligned}
$$

So we can factor out $|s|^{2(k+r-1)}$ from the first column like before, and the determinant of first $\ell \times \ell$ block is

$$
\begin{aligned}
\det &= AC^{\ell-1}(1+\frac{E}{C}(\sum_{i=2}^{\ell}|z^i|^2)) - \sum_{i=2}^{\ell} B^2 C^{\ell-2}|z^1|^2|z^i|^2 \\
&= C^{\ell-2}[A(C+E\sum_{i=2}^{\ell}|z^i|^2) - |z^1|^2B^2\sum_{i=2}^{\ell}|z^i|^2] \\
&= C^{\ell-2}A\left[C + E\sum_{i=2}^{\ell}|z^i|^2 - \frac{A}{(1-|s|^2)^2(1+|z|^2)}|s|^2\sum_{i=2}^{\ell}|z^i|^2\right] \\
&= \frac{C^{\ell-2}}{1+|z|^2}\frac{A}{1-|s|^2} \\
&\quad \Bigg[(1-|s|^2)(\sum_{m=0}^{k}|s|^{2m} - |s|^{2(k+r)}) + (1-|s|^2)\frac{(k+r+1)|s|^{2(k+r)} - \sum_{m=0}^k(m+1)|s|^{2m}}{1+|z|^2}\sum_{i=2}^{\ell}|z^i|^2
\end{aligned}
$$

$$- \left(\frac{(k+1)|s|^{2k}}{1+|z|^2} + \frac{|s|^{2(k+r-1)}}{1+|z|^2}[(k+r)-(k+r+1)|s|^2] \right) |s|^2 \sum_{i=2}^{\ell} |z^i|^2 \Big]$$

$$= \frac{C^{\ell-2}}{1+|z|^2} \frac{A}{1-|s|^2}$$

$$\Big[(1-|s|^2)(\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)})$$

$$+ \frac{(1-|s|^2)(k+r+1)|s|^{2(k+r)} - \sum_{m=0}^{k} |s|^{2m} + (k+1)|s|^{2(k+1)}}{1+|z|^2} \sum_{i=2}^{\ell} |z^i|^2$$

$$- \left(\frac{(k+1)|s|^{2(k+!)}}{1+|z|^2} + \frac{|s|^{2(k+r)}}{1+|z|^2}[(k+r)-(k+r+1)|s|^2] \right) \sum_{i=2}^{\ell} |z^i|^2 \Big]$$

$$= \frac{C^{\ell-2}}{1+|z|^2} \frac{A}{1-|s|^2}$$

$$\Big[(1-|s|^2)(\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}) + \frac{(1-|s|^2)(k+r+1)|s|^{2(k+r)} - \sum_{m=0}^{k} |s|^{2m}}{1+|z|^2} \sum_{i=2}^{\ell} |z^i|^2$$

$$- \frac{|s|^{2(k+r)}}{1+|z|^2}[(k+r)-(k+r+1)|s|^2] \sum_{i=2}^{\ell} |z^i|^2 \Big]$$

$$= \frac{C^{\ell-2}}{1+|z|^2} \frac{A}{1-|s|^2}$$

$$\Big[(1-|s|^2)(\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}) + \frac{|s|^{2(k+r)} - \sum_{m=0}^{k} |s|^{2m}}{1+|z|^2} \sum_{i=2}^{\ell} |z^i|^2 \Big]$$

$$= \frac{C^{\ell-1}}{1+|z|^2} \frac{A}{1-|s|^2} \left[(1-|s|^2)(1+|z|^2) - \sum_{i=2}^{\ell} |z^i|^2 \right] = \frac{C^{\ell-1}}{1+|z|^2} \frac{A}{1-|s|^2} \left[1+|z|^2 - \sum_{i=1}^{\ell} |z^i|^2 \right].$$

In particular,

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{C^{n-1}}{1+|z|^2} \frac{A}{1-|s|^2} \left[1+|z|^2 - \sum_{i=1}^{n} |z^i|^2 \right] = \frac{C^{n-1}}{1+|z|^2} \frac{A}{1-|s|^2}$$

$$= C^{n-1} \frac{1}{(1+|z|^2)^2} [(k+1)|s|^{2k} + |s|^{2(k+r-1)}((k+r)-(k+r+1)|s|^2)]$$

$$= |s|^{2(k+r-1)} C^{n-1} \frac{1}{(1+|z|^2)^2} [(k+1)|s|^{2(1-r)} + (k+r) - (k+r+1)|s|^2],$$

with $(k+1)|s|^{2(1-r)} + (k+r) - (k+r+1)|s|^2 > 0$ since $|s|^2 \leq |s|^{2(1-r)} < 1$. Hence this gives a metric outside the divisor with vanishing order $|s|^{2(k+r-1)}$ on the divisor on chart U_0, with behavior like cone metric associated to $|s|^{2(k+r-1)}$.

Similarly we have to check on chart U_1, and we exchange 0 and 1 as before. Then

$$\varphi = -\sum_{m=1}^{k} \frac{1}{m} |s|^{2m} + \frac{1}{k+r} |s|^{2(k+r)}$$

$$= -\sum_{m=1}^{k} \frac{1}{m} \left(\frac{1}{1+|z|^2} \right)^{2m} + \frac{1}{k+r} \left(\frac{1}{1+|z|^2} \right)^{2(k+r)}.$$

As before we have

$$(|s|^{2k})_{i\bar{j}} = k|s|^{2k}[\frac{-\delta_{ij}}{1+|z|^2} + (k+1)\frac{\bar{z}^i z^j}{(1+|z|^2)^2}],$$

therefore we get

$$\begin{aligned} g_{i\bar{j}} + \varphi_{i\bar{j}} &= \frac{\delta_{ij}}{1+|z|^2} - \frac{\bar{z}^i z^j}{(1+|z|^2)^2} + \sum_{m=1}^{k} |s|^{2m}[\frac{\delta_{ij}}{1+|z|^2} - (m+1)\frac{\bar{z}^i z^j}{(1+|z|^2)}] \\ &\quad + |s|^{2(k+r)}[\frac{-\delta_{ij}}{1+|z|^2} + (k+r+1)\frac{\bar{z}^i z^j}{(1+|z|^2)}] \\ &= (\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)})\frac{\delta_{ij}}{1+|z|^2} \\ &\quad - (\sum_{m=0}^{k}(m+1)|s|^{2m} - (k+r+1)|s|^{2(k+r)})\frac{\bar{z}^i z^j}{(1+|z|^2)^2}. \end{aligned}$$

And using Lemma 3.3,

$$\begin{aligned} \det &= \left(\frac{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2}\right)^n \left(1 - \frac{\sum_{m=0}^{k}(m+1)|s|^{2m} - (k+r+1)|s|^{2(k+r)}}{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}\frac{|z|^2}{1+|z|^2}\right) \\ &= \left(\frac{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2}\right)^n \left(1 - \frac{\sum_{m=0}^{k}(m+1)|s|^{2m} - (k+r+1)|s|^{2(k+r)}}{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}(1-|s|^2)\right) \\ &= \left(\frac{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2}\right)^n \left(1 - \frac{\sum_{m=0}^{k} |s|^{2m} - (k+1)|s|^{2(k+1)} - (k+r+1)|s|^{2(k+r)}(1-|s|^2)}{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}\right) \\ &= \left(\frac{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2}\right)^{n-1} \frac{-|s|^{2(k+r)} + (k+1)|s|^{2(k+1)} + (k+r+1)|s|^{2(k+r)}(1-|s|^2)}{1+|z|^2} \\ &= \left(\frac{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2}\right)^{n-1} \frac{(k+r)|s|^{2(k+r)} + (k+1)|s|^{2(k+1)} - (k+r+1)|s|^{2(k+r+1)}}{1+|z|^2} > 0. \end{aligned}$$

The determinant of first $\ell \times \ell$ block can also be easily seen to be positive as the last factor should be bigger.

In this example we can see that $\varphi = -\sum_{m=1}^{k} \frac{1}{m}|s|^{2m} + \frac{1}{k+r}|s|^{2(k+r)}$ gives determinant

$$\frac{|s|^{2(k+r-1)}}{(1+|z|^2)^2}\left(\frac{\sum_{m=0}^{k} |s|^{2m} - |s|^{2(k+r)}}{1+|z|^2}\right)^{n-1} [(k+1)|s|^{2(1-r)} + (k+r) - (k+r+1)|s|^2],$$

hence we may expect that the solution φ for $|s|^{2k}, k \in \mathbb{R}$ will look like $\varphi = f + |s|^{2k}g$, where f, g are smooth function. (In other words, the solution φ is not too bad.)

3.3 Other smooth divisors

Some examples with different smooth divisors are also calculated using the computer. We simply present the result and will not do the full calculation:

In $\mathbb{C}P^2$, we pick $L = O(2)$, $s = (Z^0)^2 + (Z^1)^2 + (Z^2)^2$ and $D = \{s = 0\}$ is smooth. Then on chart U_0, $|s|^2 = \frac{|1+(z^1)^2+(z^2)^2|^2}{(1+|z|^2)^2}$. Now consider $\varphi = -\frac{1}{4}|s|^2$, then on U_0 (hence on any U_i by symmetry):

$$\det(\omega + i\partial\bar{\partial}\varphi) = \frac{3}{4}|s|^2\frac{(|s|^2+2)}{(1+|z|^2)^3}$$

And if we add $-\frac{3}{32}|s|^4$, $-\frac{5}{96}|s|^6$ after φ, we will get a vanishing order of $|s|^4$ and $|s|^6$.

However, if we pick $L = O(2)$, $s = (Z^0)^2 + (Z^1)^2 + (Z^2)^2 + Z^0Z^1 + Z^0Z^2 + Z^1Z^2$. Then it seems that there is no constant $C \in \mathbb{R}$ such that $\omega + C \cdot i\partial\bar{\partial}|s|^2$ will have a vanishing order of $|s|^2$.

4 Examples for Normal crossing divisors

In this section, we consider the simple normal crossing divisor consisting of hyperplanes in $\mathbb{C}P^n$.

4.1 Normal crossing degeneracy from pullback

A first suggestive example for normal crossing divisor comes from the pullback of the metric on $\mathbb{C}P^n$. For all $m \geq 2$, if we consider the map

$$\begin{array}{ccc} \mathbb{C}P^n & \xrightarrow{j} & \mathbb{C}P^n \\ [Z^0 : Z^1 : \cdots : Z^n] & \mapsto & [(Z^0)^m : (Z^1)^m : \cdots : (Z^n)^m], \end{array}$$

then since the tangent map is not surjective on the normal crossing divisor $\{Z^0 = 0\} \cup \cdots \cup \{Z^n = 0\}$, the pullback of Fubini-Study metric via j will be singular on it.

More precisely, the pullback of a Kähler metric satisfies $j^*\omega = j^*i\partial\bar{\partial}\varphi = i\partial\bar{\partial}(\varphi \circ j)$, where φ is a local potential of ω. Hence on chart $U_0 = \{Z^0 \neq 0\}$ with coordinates $(z^1, \ldots, z^n)$,

$$\begin{aligned} g'_{i\bar{j}} = (j^*g_{FS})_{i\bar{j}} = \partial_i\bar{\partial}_j \log(1 + \sum_i |z^i|^{2m}) &= \partial_i\left(\frac{m|z^j|^{2(m-1)}z^j}{1+\sum|z^i|^{2m}}\right) \\ = \frac{\delta_{ij}m^2|z^i|^{2(m-1)}}{1+\sum|z^i|^{2m}} &- \frac{m^2|z^i|^{2(m-1)}|z^j|^{2(m-1)}\bar{z}^i z^j}{(1+\sum|z^i|^{2m})^2}, \end{aligned}$$

and the determinant is

$$\det g'_{i\bar{j}} = \det\left(\frac{\delta_{ij}m^2|z^i|^{2(m-1)}}{1+\sum|z^i|^{2m}}\right)\left(1 - \frac{1}{1+\sum|z^i|^{2m}}\sum|z^i|^{2m}\right) = \frac{m^{2n}\prod_i|z^i|^{2(m-1)}}{(1+\sum|z^i|^{2m})^{n+1}},$$

which will have same vanishing order $|z^i|^{2(m-1)}$ on each divisor $\{Z^i = 0\}$. And this corresponds to the case where $L = \mathcal{O}(1)^{\otimes(m-1)(n+1)}$ with induced hermitian metric from $\mathcal{O}(1)$ and $s = \prod_{i=0}^n (s_i)^{m-1} \in \Gamma(L)$ where $s_i \in \Gamma(\mathcal{O}(1))$ is the section that has vanishing order 1 on divisor $\{Z^i = 0\}$

Now, since $\omega = i\partial\bar{\partial}\log\sum|Z^i|^2$ and $j^*\omega = i\partial\bar{\partial}\log\sum|Z^i|^{2m}$, for the difference of potential to be globally defined, we need it to be homogeneous. In other words, consider

$$\omega - \frac{1}{m}j^*\omega = i\partial\bar{\partial}(\log\sum|Z^i|^2 - \frac{1}{m}\log\sum|Z^i|^{2m}) = i\partial\bar{\partial}\log\frac{\sum|Z^i|^2}{(\sum|Z^i|^{2m})^{\frac{1}{m}}},$$

then the difference will be $\varphi = -\log\frac{\sum|Z^i|^2}{(\sum|Z^i|^{2m})^{\frac{1}{m}}} \in C^\infty(M)$.

Remark 4.1 (Local behavior of degenerate metric)**.** The pullback metric on U_0 looks like

$$g'_{i\bar{j}} = \begin{pmatrix} \frac{m^2|z^1|^{2(m-1)}(1+\sum|z^i|^{2m}-|z^1|^{2m})}{(1+\sum|z^i|^{2m})^2} & \frac{-m^2|z^1|^{2(m-1)}|z^2|^{2(m-1)}\bar{z}^1z^2}{(1+\sum|z^i|^{2m})^2} & \cdots & \frac{-m^2|z^1|^{2(m-1)}|z^n|^{2(m-1)}\bar{z}^1z^n}{(1+\sum|z^i|^{2m})^2} \\ \frac{-m^2|z^1|^{2(m-1)}|z^2|^{2(m-1)}z^1\bar{z}^2}{(1+\sum|z^i|^{2m})^2} & \frac{m^2|z^2|^{2(m-1)}(1+\sum|z^i|^{2m}-|z^2|^{2m})}{(1+\sum|z^i|^{2m})^2} & & \\ \vdots & & \ddots & \\ \frac{-m^2|z^1|^{2(m-1)}|z^n|^{2(m-1)}z^1\bar{z}^n}{(1+\sum|z^i|^{2m})^2} & & & \frac{m^2|z^n|^{2(m-1)}(1+\sum|z^i|^{2m}-|z^n|^{2m})}{(1+\sum|z^i|^{2m})^2} \end{pmatrix}.$$

Like before, we can factor out $|z^i|^{2(m-1)}$ from each column, which contributes to the final degeneration of the determinant.

And from the above example, we may guess that the behavior of metric with degeneration $\prod_{i=1}^n |z^i|^{2r_i}$ will be:

$$\begin{pmatrix} |z^1|^{2r_1} & |z^1|^{2r_1}\cdot|z^2|^{2r_2} & |z^1|^{2r_1}\cdot|z^3|^{2r_3} & \cdots \\ |z^1|^{2r_1}\cdot|z^2|^{2r_2} & |z^2|^{2r_2} & |z^2|^{2r_2}\cdot|z^3|^{2r_3} & \cdots \\ |z^1|^{2r_1}\cdot|z^3|^{2r_3} & |z^2|^{2r_2}\cdot|z^3|^{2r_3} & |z^3|^{2r_3} & \cdots \\ & \vdots & & \end{pmatrix} \tag{19}$$

under suitable local coordinates such that $\{\prod (z^i)^{r_i} = 0\}$ defines the divisor.

4.2 Local construction with normal crossing degeneracy

Following the above example, we continue to consider the metric of Fubini-Study type. For more general normal crossing divisor that has different order on each part, we may take locally in coordinates $(z^1, \ldots, z^n)$

$$\omega = i\partial\bar{\partial}\log(1 + \sum_{i=1}^n |z^i|^{2(r_i+1)}),$$

with $r_i \neq -1 \in \mathbb{R}$ being the vanishing order we want on $\{z^i = 0\}$.

Then

$$\begin{aligned} g_{i\bar{j}} = \log(1 + \sum_{i=1}^n |z^i|^{2(r_i+1)})_{i\bar{j}} &= \left(\frac{(r_i+1)|z^i|^{2r_i}\bar{z}^i}{1+\sum_{i=1}^n |z^i|^{2(r_i+1)}}\right)_{\bar{j}} \\ &= \frac{(r_i+1)^2|z^i|^{2r_i}}{1+\sum_{i=1}^n |z^i|^{2(r_i+1)}}\delta_{ij} - \frac{(r_i+1)(r_j+1)|z^i|^{2r_i}|z^j|^{2r_j}\bar{z}^i z^j}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^2} \end{aligned}$$

and

$$\begin{aligned} \det g_{i\bar{j}} &= \frac{\prod_{i=1}^n (r_i+1)^2|z^i|^{2r_i}}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^n}(1 - \frac{\sum_{i=1}^n |z^i|^{2(r_i+1)}}{1+\sum_{i=1}^n |z^i|^{2(r_i+1)}}) \\ &= \frac{\prod_{i=1}^n (r_i+1)^2|z^i|^{2r_i}}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^{n+1}}, \end{aligned}$$

which has vanishing order $\prod |z^i|^{2r_i}$ as we hope.

For positive definiteness, we can see that the determinant of first $\ell \times \ell$ square is

$$\frac{\prod_{i=1}^\ell (r_i+1)^2|z^i|^{2r_i}}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^\ell}(1 - \frac{\sum_{i=1}^\ell |z^i|^{2(r_i+1)}}{1+\sum_{i=1}^n |z^i|^{2(r_i+1)}}) = \left(\prod_{i=1}^\ell (r_i+1)^2|z^i|^{2r_i}\right)\frac{1+\sum_{i=\ell+1}^n |z^i|^{2r_i+1}}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^{\ell+1}} \geq 0.$$

So the metric is positive definite outside the divisor.

If we let $G = 1 + \sum_{i=1}^n |z^i|^{2(r_i+1)}$, then

$$\begin{aligned} g_{i\bar{j}} &= (\log G)_{i\bar{j}} = \frac{G_{i\bar{j}}}{G} - \frac{G_i G_{\bar{j}}}{G^2} \\ &= \frac{(r_i+1)^2|z^i|^{2r_i}}{1+\sum_{i=1}^n |z^i|^{2(r_i+1)}}\delta_{ij} - \frac{(r_i+1)(r_j+1)|z^i|^{2r_i}|z^j|^{2r_j}\bar{z}^i z^j}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^2} \\ &= \begin{cases} |z^i|^{2r_i}\frac{(r_i+1)^2(1+\sum_{j\neq i}|z^j|^{2(r_j+1)})}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^2} & i = j \\ -|z^i|^{2r_i}|z^j|^{2r_j}\frac{(r_i+1)(r_j+1)\bar{z}^i z^j}{(1+\sum_{i=1}^n |z^i|^{2(r_i+1)})^2} & i \neq j \end{cases} \\ &= \begin{pmatrix} |z^1|^{2r_1}\frac{(r_1+1)^2(G-|z^1|^{2(r_1+1)})}{G^2} & -|z^1|^{2r_1}|z^2|^{2r_2}\frac{(r_1+1)(r_2+1)\bar{z}^1 z^2}{G^2} & \cdots \\ -|z^2|^{2r_2}|z^1|^{2r_1}\frac{(r_2+1)(r_1+1)\bar{z}^2 z^1}{G^2} & |z^2|^{2r_2}\frac{(r_2+1)^2(G-|z^2|^{2(r_2+1)})}{G^2} & \cdots \\ \vdots & \vdots & \end{pmatrix} \end{aligned}$$

as guessed in Equation (19).

From this we also see that the degenerate metric is still a metric when restricted to the smooth part of the divisor.

Remark 4.2. We also have a simpler local construction by considering

$$(\sum_{i=1}^n |z^i|^{2(r_i+1)})_{i\bar{j}} = \delta_{ij}(r_i+1)^2|z^i|^{r_i}.$$

This gives a similar degeneration, and can be globalized to normal crossing divisor in $(\mathbb{C}P^1)^n$.

4.3 Globalization

With a little tweaking, we can make the above example defined globally. Our main result for this section is:

Proposition 4.3. *For $r_i \in \mathbb{R}_{>0}$, $1 \leq i \leq n$, consider*

$$\omega = i\partial\bar{\partial}\log(|Z^0|^{2m} + \sum_{i=1}^n |Z^i|^{2(r_i+1)}|Z^0|^{2(m-r_i-1)} + \sum_{i=1}^n |Z^i|^{2m}),$$

where $m \in \mathbb{N}$ is greater than or equal to $r_i + 1$ for all $1 \leq i \leq n$. Then ω is well defined on whole $\mathbb{C}P^n$ and determines a metric outside of the divisor $\{Z^0 = 0\} \cup (\bigcup_{r^i>0}\{Z^i = 0\})$, with vanishing order $|z^i|^{2r_i}$ on $\{Z^i = 0\}$ for each $i \geq 1$. However, although we can create any vanishing order r_i on each $\{Z^i = 0\}$ for $i \neq 0$. The vanishing behavior on $\{Z^0 = 0\}$, is determined by other $\{r_i\}_{i=1}^n$ in this construction.

And the difference of

$$\omega_{FS} - \frac{1}{m}\omega = i\partial\bar{\partial}\log\frac{\sum_{i=0}^n |Z^i|^2}{(|Z^0|^{2m} + \sum_{i=1}^n |Z^i|^{2(r_i+1)}|Z^0|^{2(m-r_i-1)} + \sum_{i=1}^n |Z^i|^{2m})^{\frac{1}{m}}} = i\partial\bar{\partial}\varphi$$

gives a globally defined potential φ.

For

$$\omega = i\partial\bar{\partial}\log(|Z^0|^{2m} + \sum_{i=1}^n |Z^i|^{2(r_i+1)}|Z^0|^{2(m-r_i-1)} + \sum_{i=1}^n |Z^i|^{2m}),$$

on chart U_0,

$$
\begin{aligned}
g_{i\bar{j}} &= \log(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})_{i\bar{j}} \\
&= \left(\frac{(r_i+1)|z^i|^{2r_i}\bar{z}^i+m|z^i|^{2(m-1)}\bar{z}^i}{1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m}}\right)_{\bar{j}} \\
&= \frac{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m}}\delta_{ij} - \frac{((r_i+1)|z^i|^{2r_i}+m|z^i|^{2(m-1)})*(j\text{ part})}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^2}\bar{z}^i z^j.
\end{aligned}
$$

Then using lemma 3.3

$$
\begin{aligned}
\det g_{i\bar{j}} &= \frac{\prod_{i=1}^{n}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^n} \\
&\quad \left(1-\sum_{i=1}^{n}\frac{|z^i|^2}{1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m}}\frac{((r_i+1)|z^i|^{2r_i}+m|z^i|^{2(m-1)})^2}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right) \\
&= \frac{\prod_{i=1}^{n}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^{n+1}} \\
&\quad \left(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m}-\sum_{i=1}^{n}|z^i|^2\frac{((r_i+1)|z^i|^{2r_i}+m|z^i|^{2(m-1)})^2}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right) \\
&= \frac{\prod_{i=1}^{n}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^{n+1}} \\
&\quad \left(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+|z^i|^{2m}-|z^i|^2\frac{(r_i+1)^2(|z^i|^{2r_i})^2+m^2(|z^i|^{2(m-1)})^2+2(r_i+1)m|z^i|^{2(m+r_i+1)}}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right) \\
&= \frac{\prod_{i=1}^{n}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^{n+1}} \\
&\quad \left(1+\sum_{i=1}^{n}|z^i|^2\frac{m^2|z^i|^{2(m+r_i+1)}+(r_i+1)^2|z^i|^{2(m+r_i-1)}-2(r_i+1)m|z^i|^{2(m+r_i+1)}}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right) \\
&= \frac{\prod_{i=1}^{n}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^{n+1}}\left(1+\sum_{i=1}^{n}|z^i|^2\frac{(m-r_i-1)^2|z^i|^{2(m+r_i+1)}}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right) \\
&= \prod_{i=1}^{n}|z^i|^{2r_i}\frac{\prod_{i=1}^{n}(r_i+1)^2+m^2|z^i|^{2(m-r_i-1)}}{(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m})^{n+1}}\left(1+\sum_{i=1}^{n}|z^i|^2\frac{(m-r_i-1)^2|z^i|^{2(m+r_i+1)}}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right)
\end{aligned}
$$

For the positive definiteness, the determinant of the first $\ell \times \ell$ block is

$$\begin{aligned}
&\frac{\prod_{i=1}^{\ell}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^n|z^i|^{2(r_i+1)}+\sum_{i=1}^n|z^i|^{2m})^{\ell}}\\
&\left(1-\sum_{i=1}^{\ell}\frac{|z^i|^2}{1+\sum_{i=1}^n|z^i|^{2(r_i+1)}+\sum_{i=1}^n|z^i|^{2m}}\frac{((r_i+1)|z^i|^{2r_i}+m|z^i|^{2(m-1)})^2}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right)\\
=&\frac{\prod_{i=1}^{\ell}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^n|z^i|^{2(r_i+1)}+\sum_{i=1}^n|z^i|^{2m})^{\ell+1}}\\
&\left(1+\sum_{i=1}^{n}|z^i|^{2(r_i+1)}+\sum_{i=1}^{n}|z^i|^{2m}-\sum_{i=1}^{\ell}|z^i|^2\frac{((r_i+1)|z^i|^{2r_i}+m|z^i|^{2(m-1)})^2}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}\right)\\
=&\frac{\prod_{i=1}^{\ell}(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}{(1+\sum_{i=1}^n|z^i|^{2(r_i+1)}+\sum_{i=1}^n|z^i|^{2m})^{\ell+1}}\\
&\left(1+\sum_{i=1}^{\ell}|z^i|^2\frac{(m-r_i-1)^2|z^i|^{2(m+r_i+1)}}{(r_i+1)^2|z^i|^{2r_i}+m^2|z^i|^{2(m-1)}}+\sum_{i=\ell+1}^{n}|z^i|^{2(r_i+1)}+|z^i|^{2m}\right),
\end{aligned}$$

which is positive outside the divisor.

Remark 4.4. Although we can assign any degree on the divisor $\{Z^i=0\}$ for $1 \le i \le n$. This will certainly create some order on the divisor $\{Z^0=0\}$ (which should be $|z^0|^{\min_i 2(m-r_i-2)}$). We hope to fully calculate this in the future.

5 A Proposal on the openness of degenerate complex Monge–Ampère equations

With the above examples as a starting point, we can start to set up the continuity method. For reference, we first review how it is done in [1].

To solve the complex Monge–Ampère equation

$$(\omega+i\partial\bar\partial\varphi)^n=e^F\omega^n \tag{20}$$

we deform it through the following equation

$$(\omega+i\partial\bar\partial\varphi)^n=C_te^{tF}\omega^n,\ t\in[0,1].$$

where $C_t=\mathrm{vol}(M)/\int e^{tF}\omega^n$ is constant for each t which makes Equation (2) hold.

Then we consider the set

$$\mathcal{S}=\{t\in[0,1]|\text{ the equation }(\omega+i\partial\bar\partial\varphi)^n=C_te^{tF}\omega^n\text{ has solution }\varphi\in C^{k+1,\alpha}(M)\}.$$

We have $0\in\mathcal{S}$, since at $t=0$ the equation is just

$$(\omega+i\partial\bar\partial\varphi)^n=\omega^n,$$

hence $\varphi=0$ is a solution. If we can show that S is both open and closed, then since $[0,1]$ is connected, we must have $\mathcal{S}=[0,1]$. In particular, $1\in\mathcal{S}$, so Equation (20) must have a solution in $C^{k+1,\alpha}(M)$.

For the openness, we use the inverse function theorem for Banach space. Let

$$\Theta=\{\varphi\in C^{k+1,\alpha}(M)|(g_{i\bar j}+\varphi_{i\bar j})>0,\int\varphi=0\},$$

$$\mathcal{B} = \{f \in C^{k-1,\alpha}(M) | \int f\omega^n = \mathrm{vol}(M)\}.$$

Then Θ is an open set in Banach space $C^{k+1,\alpha}(M)$ (since eigenvalues will have lower bound), and $\mathcal{B}$ is an affine plane in $C^{k-1,\alpha}(M)$. We have a map $G : \Theta \to \mathcal{B}$

$$G(\varphi) = \frac{\det(g_{i\bar{j}} + \partial^2_{i\bar{j}}\varphi)}{\det g_{i\bar{j}}}.$$

With differential at φ is

$$dG(\delta\varphi) = \delta G(\varphi) = \frac{\delta \det(g_{i\bar{j}} + \partial^2_{i\bar{j}}\varphi)}{\det g_{i\bar{j}}} = \frac{\det(g_{i\bar{j}} + \partial^2_{i\bar{j}}\varphi)}{\det g_{i\bar{j}}}\Delta_{g_\varphi}(\delta\varphi).$$

Hence, since the tangent space of $\mathcal{B}$ is $\{f \in C^{k-1,\alpha}(M) | \int f\omega^n = 0\}$. And for

$$dG(\delta\varphi) = \frac{\det(g_{i\bar{j}} + \partial^2_{i\bar{j}}\varphi)}{\det g_{i\bar{j}}}\Delta_{g_\varphi}(\delta\varphi) = f$$

to have weak solution, we have $\Delta_{g_\varphi}(\delta\varphi)\det(g_{i\bar{j}}+\varphi_{i\bar{j}}) = f \det g_{i\bar{j}}$, so $\int f\omega^n = \int \Delta_{g_\varphi}(\delta\varphi)(\omega + i\partial\bar{\partial}\varphi)^n = 0$, which is precisely the requirement for tangent space of $\mathcal{B}$. Thus we have a weak solution φ, and by usual Schauder's estimates, $\varphi \in C^{k+1,\alpha}(M)$. φ is also unique if we require $\int \varphi = 0$, thus the differential is invertible and we can apply the inverse function theorem for Banach space. This shows that if we have a solution to Equation (20), then for a small change on the right-hand side, the equation is still solvable. In particular, this proves the openness for $\mathcal{S}$.

We won't talk about closeness here, instead we will try to work on the openness for degenerate metric. In the degenerate case, for any $|s|^{2k}$, it is not clear why there should be a smooth solution to any Equation (4). This is why we have to construct some examples and hope to work on the general existence in the future. With these examples as a starting point, we can try to prove the openness near these metric. This part has been treated in [2], in the case of $|s|^2$ with simple zeros on the smooth divisor. We hope to deal with general smooth divisors with any order $|s|^{2k}$ based on our constructions.

As in the nondegenerate case, we consider the same $G : C^{\ell+1,\alpha}(M) \to \mathcal{B}$. We now look at the map near the $\omega' = \omega + i\partial\bar{\partial}\varphi$ we constructed in Section 3, which is degenerate on the divisor. Then the differential dG at φ will be

$$dG(\delta\varphi) = \delta\frac{\det(g+\varphi)}{\det g} = \frac{A^{i\bar{j}}}{\det g}(\delta\varphi)_{i\bar{j}} = \frac{\det g'}{\det g}\Delta_{g'}(\delta\varphi), \tag{21}$$

where $A^{i\bar{j}}$ is the (i,j)-cofactor of $(g_{i\bar{j}} + \varphi_{i\bar{j}})$. And because the metric is degenerate on the divisor, $\Delta_{g'}$ is only well defined outside the divisor.

So for the invertiblilty of dG at φ, we have to solve the degenerate Laplace equation associated to conical metric

$$\det g'\Delta_{g'}(\delta\varphi) = f \det g.$$

for any $f \in C^{\ell-1,\alpha}(M)$ that satisfies $\int f\omega^n = 0$. As treated in [2], We hope to use the result of [6] about Hodge theory in Riemannian manifold with non isolated conical singularity, but this requires some further investigation.

Suppose that we can solve the degenerate Laplace equation; then we have to show that the solution belongs to $C^{\ell+1,\alpha}(M)$. For this, we need to develop Schauder's estimates associated with the conical metric. Since the metric is nondegenerate outside the divisor, we only have to check the regularity on the divisor.

Now we investigate the case where φ is the one we constructed, then we can find coordinates near the divisor such the metric is of the form as in **??**, i.e.

$$g'_{i\bar{j}} = \begin{pmatrix} |z^1|^{2r_1} & |z^1|^{2r_1}\cdot|z^2|^{2r_2} & |z^1|^{2r_1}\cdot|z^3|^{2r_3} & \cdots \\ |z^1|^{2r_1}\cdot|z^2|^{2r_2} & |z^2|^{2r_2} & |z^2|^{2r_2}\cdot|z^3|^{2r_3} & \cdots \\ |z^1|^{2r_1}\cdot|z^3|^{2r_3} & |z^2|^{2r_2}\cdot|z^3|^{2r_3} & |z^3|^{2r_3} & \cdots \\ & \vdots & & \end{pmatrix}.$$

For simplicity we only consider the smooth divisor first, that is

$$g'_{i\bar{j}} = \begin{pmatrix} |s|^{2k} & |s|^{2k} & \cdots \\ |s|^{2k} & * & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then we can observe that $A^{i\bar{j}}$ will have a factor of $|s|^{2k}$ unless $i = j = 1$. So for $\Delta_{g'}(\delta\varphi) = g'^{i\bar{j}}(\delta\varphi)_{i\bar{j}} = \frac{A^{i\bar{j}}}{|s|^{2k}e^F \det g}(\delta\varphi)_{i\bar{j}}$, we can see that only the term $g'^{1\bar{1}}$ will have a $\frac{1}{|z^1|^{2k}}$ in it, the remaining $g'^{i\bar{j}}$ will be smooth.

Another possible way to get the regularity is based on the observation that $g'_{i\bar{j}}$ is the pullback of $\tilde{g}_{i\bar{j}}$ under the map

$$(z^1, z^2, \ldots) \to (w = (z^1)^{k+1}, z^2, \ldots).$$

Where

$$\tilde{g}_{i\bar{j}} = \begin{pmatrix} \frac{1}{(k+1)(1+|w|^{\frac{2}{k+1}}+|z'|^2)^{k+1}}(1-|s|^2) & -\frac{\bar{w}}{(1+|w|^{\frac{2}{k+1}}+|z'|^2)^{k+2}}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \\ -\frac{w}{(1+|w|^{\frac{2}{k+1}}+|z'|^2)^{k+2}}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix} & \frac{\sum_{m=0}^{k}|s|^{2m}}{1+|w|^{\frac{2}{k+1}}+|z'|^2} I_{n-1} - \frac{\sum_{m=0}^{k}(m+1)|s|^{2m}}{(1+|w|^{\frac{2}{k+1}}+|z'|^2)^2}\begin{pmatrix} \bar{z}^2 \\ \vdots \\ \bar{z}^n \end{pmatrix}\begin{pmatrix} z^2 & \cdots & z^n \end{pmatrix} \end{pmatrix},$$

with $|z'|^2 = \sum_{i\geq 2}|z^i|^2$ and $|s|^2 = \frac{|w|^{\frac{2}{k+1}}}{1+|w|^{\frac{2}{k+1}}+|z'|^2}$. It is well defined and nondegenerate in a neighborhood of $\{w = 0\}$, so it is likely that we may use the usual Schauder's estimates.

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